Memorial University of Newfoundland **Winter Semester 2022**
Yorck Sommerhäuser **MATH 4310: Sheet 3**

Complex Function Theory

Problem 1: Suppose that $\gamma : [t_0, t_1] \to \mathbb{C}$ is a smooth closed curve that does not pass through the origin. For $t_2 \in [t_0, t_1]$, define $t_3 := t_2 + (t_1 - t_0)$ and consider the curve $\delta : [t_2, t_3] \to \mathbb{C}$ defined by

$$\delta(t) := \begin{cases} \gamma(t) & : t \leq t_1 \\ \gamma(t - (t_1 - t_0)) & : t > t_1 \end{cases}$$

which traces out the same path as γ, but begins at the later time t_2. Show that the winding numbers $W(\gamma)$ and $W(\delta)$ are equal.

(Hint: Recall that, if $\gamma(t)/|\gamma(t)| = a(t) + ib(t)$, then

$$\phi(t) := \phi_0 + \int_{t_0}^{t} a(\tau)b'(\tau) - b(\tau)a'(\tau)d\tau$$

where ϕ_0 is required to satisfy $a(t_0) = \cos(\phi_0)$ and $b(t_0) = \sin(\phi_0)$, has the property that $a(t) = \cos(\phi(t))$ and $b(t) = \sin(\phi(t))$ for all $t \in [t_0, t_1]$. The winding number of γ is $W(\gamma) = \frac{1}{2\pi}(\phi(t_1) - \phi(t_0))$. To solve the problem, relate these quantities to the corresponding quantities for the curve δ.) (25 points)

Problem 2: Suppose that $\gamma : [t_0, t_1] \to \mathbb{C}$ is a smooth closed curve that does not pass through the origin, and suppose that it also does not intersect the ray

$$\{re^{i\alpha} | r \geq 0\}$$

where α is a given, fixed angle. Show that $W(\gamma) = 0$.

(Hint: Use the intermediate value theorem. This is a reformulation of Exercise 3 for Section 80 in the textbook, which contains additional hints.) (25 points)

Problem 3: Suppose that D is a bounded domain in the complex plane whose boundary is parametrized by a simple smooth closed curve C. Suppose that f is a function that is analytic at all points of C and D, and that f is not identically zero. Show that f has at most finitely many zeros in D.

(Hint: This is a variant of Exercise 10 for Section 69 or Exercise 4 for Section 80 in the textbook, which contains additional hints. Note that a domain is connected in the sense described in Section 10.) (25 points)

Problem 4: Suppose that D is a bounded domain in the complex plane whose boundary is parametrized by a simple smooth closed curve C. Suppose that f is a function that is meromorphic on D and analytic on all points of C. Show that f has at most finitely many poles in D.

(Hint: This is a variant of Exercise 11 for Section 69 in the textbook, which contains additional hints. Note that by definition a pole is an isolated singular point, and discuss whether an accumulation point of the set of poles can be a pole itself.) (25 points)

Due date: Monday, February 7, 2022. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.