Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 4230: Sheet 9**

Differential Geometry

Problem 1: We have seen in Problem 4 on Sheet 7 that the pseudosphere S can be parametrised by

$$f\colon]0,2\pi[\times]1,\infty[\to S,\ (\varphi,w)\mapsto f(\varphi,w):=\left(\frac{1}{w}\cos(\varphi),\frac{1}{w}\sin(\varphi),\tau\left(\frac{1}{w}\right)\right)$$

where the function τ was introduced in Problem 3 on Sheet 6. Suppose that

$$\alpha\colon I:=\,]a,b[\,\to S,\ s\mapsto\alpha(s):=f(\alpha_1(s),\alpha_2(s))$$

is a geodesic that is parametrised by arc-length and which is not a meridian.

1. From the assumption that $|\alpha'(s)|=1$, deduce that
$$\alpha_1'(s)^2+\alpha_2'(s)^2=\alpha_2(s)^2$$
(9 points)

2. From Clairaut's relation in the form given at the bottom of page 259 in the textbook, deduce that that there is a constant c such that
$$\alpha_1'(s)=c\,\alpha_2(s)^2$$
and show that c is not zero. (8 points)

3. Show that $\alpha_2'(s)^2=\alpha_2(s)^2(1-c^2\,\alpha_2(s)^2)$. (8 points)

Problem 2: Continuing the previous problem, we now assume that $\alpha_2'(s)\neq 0$ for all $s\in I$. Then the function $\alpha_2\colon I\to\mathbb{R},\ s\mapsto\alpha_2(s)$ can be inverted, so that there is an inverse function $\tilde{s}:\alpha_2(I)\to I$ with the property that $\tilde{s}(\alpha_2(s))=s$ and $\alpha_2(\tilde{s}(w))=w$. We define $\tilde{\varphi}(w):=\alpha_1(\tilde{s}(w))$.

1. Show that
$$\tilde{\varphi}'(w)=\pm\frac{cw}{\sqrt{1-c^2w^2}}$$
(9 points)

2. Show that there is a constant φ_0 with the property that
$$(\tilde{\varphi}(w)-\varphi_0)^2+w^2=\frac{1}{c^2}$$
(8 points)

3. Show that $(\alpha_1(s)-\varphi_0)^2+\alpha_2(s)^2=\frac{1}{c^2}$ for all $s\in I$. So the curve $(\alpha_1(s),\alpha_2(s))$ lies on a circle with radius $1/c$ and center $(\varphi_0,0)$ in the φ,w-plane. (8 points)

Problem 3: Under the hypotheses of Problem 2, show that the length L of α between two points $\alpha(s_1)$ and $\alpha(s_2)$ is

$$L = \ln|\tan(\theta_2/2)| - \ln|\tan(\theta_1/2)|$$

where $\alpha_1(s_1) = \varphi_0 + \frac{1}{c}\cos(\theta_1)$ and $\alpha_1(s_2) = \varphi_0 + \frac{1}{c}\cos(\theta_2)$. (25 points)

Problem 4: Suppose that S is a regular surface that is oriented via the unit normal vector field N, and that $\alpha : I \to S$ is a differentiable curve that is parametrised by arc-length. Recall that the normal curvature $k_n(s)$ and the geodesic curvature $k_g(s)$ of α are defined by the equation

$$\alpha''(s) = k_n(s)N(\alpha(s)) + k_g(s)(N(\alpha(s)) \times \alpha'(s))$$

Show that α is a geodesic if and only if $k_g(s) = 0$ for all $s \in I$. (25 points)

Due date: Thursday, April 1, 2021. Write your solution on letter-sized paper, scan it and submit it to the assignment box for Sheet 9 in the Brightspace site for this course. Begin your solution with a cover sheet. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.