Differential Geometry

Problem 1:

1. Compute the Gaussian curvature of the unit sphere. (13 points)
2. Show that no open set in the unit sphere can be isometric to an open set in the xy-plane. (12 points)

(Hint: Compare Problem 4 on Sheet 6 and the examples in Section 3-2 of the textbook.)

Problem 2: For a parametrisation of a regular surface, let $(g_{ij}(u,v))_{i,j=1,2}$ be the first fundamental form, so that $g_{11}(u,v) = E(u,v)$, $g_{12} = g_{21} = F$, and $g_{22} = G$, where here and in the sequel we suppress the dependence on $(u,v) = (u_1,u_2)$. If the inverse matrix is denoted by $(g^{ij})_{i,j=1,2}$, show that

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l}\right)$$

(25 points)

(Hint: Proceed as in the derivation of Equation (2) on page 235 of Section 4-3 of the textbook.)

Problem 3: The unit sphere has the parametrisation

$$f(\varphi,\theta) = (\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta))$$

In this parametrisation, find all eight Christoffel symbols $\Gamma_{ij}^k(\varphi,\theta)$, for all index values $i,j,k = 1,2$. (25 points)

Problem 4: With the notation from Problem 3 and Problem 4 on Sheet 6, the pseudosphere can also be parametrised by

$$g\colon]0,2\pi[\times]1,\infty[,\ (\varphi,w) \mapsto g(\varphi,w) := \left(\frac{1}{w}\cos(\varphi), \frac{1}{w}\sin(\varphi), \tau\left(\frac{1}{w}\right)\right)$$

1. Find the coefficients E, F, and G of the first fundamental form in this parametrisation. (10 points)
2. Find the eight Christoffel symbols Γ_{ij}^k for all index values $i,j,k = 1,2$ in this parametrisation. (15 points)

Due date: Thursday, March 18, 2021. Write your solution on letter-sized paper, scan it and submit it to the assignment box for Sheet 7 in the Brightspace site for this course. Begin your solution with a cover sheet. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.