**Memorial University of Newfoundland** **Winter Semester 2021**
**Yorck Sommerhäuser** **MATH 4230: Sheet 6**

# Differential Geometry

**Problem 1:** Suppose that $g : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is an arbitrarily often differentiable function that satisfies $g(0) = 0$, $g'(0) = 0$, but $g''(0) > 0$. Show that there is a positive number $\delta < \varepsilon$ such that $g(x) > 0$ for all $x$ with $0 < |x| < \delta$. (25 points)

(Hint: Use Taylor's inequality, which is Theorem 9 in Section 11.10 of Stewart's calculus textbook.)

**Problem 2:** Suppose that $S$ is a regular oriented surface and that $p \in S$ is a hyperbolic point. Show that, for every subset $V \subset S$ that contains $p$ and is open in $S$, there are points $q, r \in V$ that lie on opposite sides of the tangent plane $T_pS$. (25 points)

(Hint: If $N(p)$ is the normal vector at $p$ and $\alpha$ is an arbitrarily often differentiable curve with $\alpha(0) = p$, consider the function $g(t) := \langle \alpha(t) - p, N(p) \rangle$ and apply Problem 1.)

**Problem 3:** Consider the function $\tau \colon ]0,1[ \to \mathbb{R}$ defined by

$$\tau(x) := \sqrt{1-x^2} - \operatorname{arcosh}(1/x)$$

where arcosh denotes the area hyperbolic cosine. i.e., the inverse function of the hyperbolic cosine.

1. Show that $\tau'(x) := \dfrac{\sqrt{1-x^2}}{x}$ (12 points)
2. For the plane curve $\alpha(x) = (x, \tau(x))$, let $q$ be the point where the tangent line to $\alpha$, i.e., the line through $p := \alpha(x)$ with direction $\alpha'(x)$, meets the $y$-axis. Show that the distance from $p$ to $q$ is 1. (13 points)

(Remark: The curve $\alpha$ is known as the tractrix.)

**Problem 4:** The surface of revolution that arises by rotating the tractrix around the $z$-axis is called the pseudosphere. With the notation of Problem 3, it has the parametrisation

$$f \colon ]0,1[ \times ]0,2\pi[,\ (x,\varphi) \mapsto f(x,\varphi) := (x\cos(\varphi), x\sin(\varphi), \tau(x))$$

Show that the Gaussian curvature of the pseudosphere is constantly equal to $-1$. (25 points)

Due date: Tuesday, March 9, 2021. Write your solution on letter-sized paper, scan it and submit it to the assignment box for Sheet 6 in the Brightspace site for this course. Begin your solution with a cover sheet. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.