Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 4230: Sheet 5**

Differential Geometry

Problem 1: The cylinder

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

contains the point $p = (1, 0, 0)$, and at this point, the normal vector is also equal to $N(p) = (1, 0, 0)$.

1. Suppose that m is a real number, and consider the plane

$$P := \{(x, y, z) \in \mathbb{R}^3 \mid z = my\}$$

Give a parametrisation of the corresponding normal section, i.e., the curve $S \cap P$, using trigonometric functions. (13 points)

2. Compute the curvature of this curve at the point p (which is equal to its normal curvature). (10 points)

3. Find the maximum value of the normal curvature as m varies. (2 points)

(Hint: This problem should be compared with Example 7 in Section 3-2 on page 146 of the textbook. To compute the curvature of the curve in Part 2, you can use the formula in Exercise 12 in Section 1-5 on page 26 of the textbook, which we proved in class.)

Problem 2: Suppose that $a > r$ are positive numbers.

1. Find the length of the curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$ given by

$$\alpha(t) = ((a + r/2) \cos(t), (a + r/2) \sin(t), r\sqrt{3}/2)$$

by using the formula on page 6 in Section 1-3 of the textbook. (13 points)

2. The curve α lies on the torus parametrized by

$$f(u, v) = ((a + r \cos(u)) \cos(v), (a + r \cos(u)) \sin(v), r \sin(u))$$

where $0 \leq u, v \leq 2\pi$, because $\alpha(t) = f(\pi/3, t)$. Sketch the torus and indicate the curve in the picture. (12 points)

Problem 3: Consider the parametrisation f of the torus described in Problem 2.

1. Show that its first fundamental form is given by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & (a + r\cos(u))^2 \end{pmatrix}$$

(5 points)

2. For the functions $u(t) := \pi/3$ and $v(t) := t$, compute the integral

$$\int_0^{2\pi} \sqrt{E(u(t), v(t))u'(t)^2 + 2F(u(t), v(t))u'(t)v'(t) + G(u(t), v(t))v'(t)^2}dt$$

from Equation (2) on page 97 in Section 2-5 of the textbook. (10 points)

3. Explain the relation of this computation to the computation of the length of α in Problem 2. (10 points)

(Remark: Note that the first part of this problem is a repetition of what we did in class. The computation can also be found twice in the textbook, namely in Example 5 in Section 2-5 on page 101 and in Example 1 in Section 3-3 on page 159.)

Problem 4: If $S^1 \subset \mathbb{R}^2$ denotes the unit circle, then the (Clifford) torus is the set

$$T_C := S^1 \times S^1 = \{(\alpha, \beta, \rho, \sigma) \mid \alpha^2 + \beta^2 = \rho^2 + \sigma^2 = 1\} \subset \mathbb{R}^4$$

On the other hand, if $0 < r < a$, we saw in Problem 4 on Sheet 4 that the torus of revolution is

$$T_R = \{(x, y, z) \mid (a^2 - r^2 + x^2 + y^2 + z^2)^2 = 4a^2(x^2 + y^2)\} \subset \mathbb{R}^3$$

Show that

$$\tau : T_C \to T_R, \ (\alpha, \beta, \rho, \sigma) \mapsto ((a + r\alpha)\rho, (a + r\alpha)\sigma, r\beta)$$

is a bijection. (25 points)

Due date: Tuesday, March 2, 2021. Write your solution on letter-sized paper, scan it and submit it to the assignment box for Sheet 5 in the Brightspace site for this course. Begin your solution with a cover sheet. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.