Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 4230: Sheet 4**

Differential Geometry

Problem 1: Suppose that a parametrisation of a surface is the graph of a function h that is defined in an open set $U \subset \mathbb{R}^2$:

$$f(x,y) = (x, y, h(x,y))$$

1. Show that the unit normal vector is given by the formula

$$N(x,y) = \frac{1}{\sqrt{1 + (\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial y})^2}} \left(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1 \right)$$

   (Use the formula from page 89 in the textbook.) (10 points)

2. Show that the first fundamental form is given by the matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 + (\frac{\partial h}{\partial x})^2 & \frac{\partial h}{\partial x}\frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x}\frac{\partial h}{\partial y} & 1 + (\frac{\partial h}{\partial y})^2 \end{pmatrix}$$

   (15 points)

(Remark: Note that in the notation above, the argument (x,y) has been supressed: It is understood that each partial derivative of h is evaluated at (x,y).)

Problem 2: The graph of the function

$$h(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is called an elliptic paraboloid.

1. Sketch the surface, giving some indication of the meaning of the numbers a and b. (10 points)

2. Using the results of Problem 1, show that the first fundamental form is given by the matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 + \frac{4x^2}{a^4} & \frac{4xy}{a^2b^2} \\ \frac{4xy}{a^2b^2} & 1 + \frac{4y^2}{b^4} \end{pmatrix}$$

   (15 points)

(Remark: It is understood in this problem that a and b are two strictly positive constants.)

Problem 3: Another parametrisation of the elliptic paraboloid is given by

$$g(u,v) = (au\cos(v), bu\sin(v), u^2)$$

Show that the first fundamental form is given by the matrix

$$\begin{pmatrix} E' & F' \\ F' & G' \end{pmatrix} = \begin{pmatrix} a^2\cos^2(v) + b^2\sin^2(v) + 4u^2 & (b^2-a^2)u\sin(v)\cos(v) \\ (b^2-a^2)u\sin(v)\cos(v) & a^2u^2\sin^2(v) + b^2u^2\cos^2(v) \end{pmatrix}$$

(25 points)

Problem 4: Suppose that $a > r$ are positive numbers. We have seen in class, and it is stated in Example 5 of Section 2-5 on page 101 of the textbook, that the parametric equations

$$x = (a + r\cos(u))\cos(v) \qquad y = (a + r\cos(u))\sin(v) \qquad z = r\sin(u)$$

describe a torus, where $0 \le u, v < 2\pi$.

1. Show that the points on the torus satisfy the equation

$$(a^2 - r^2 + x^2 + y^2 + z^2)^2 = 4a^2(x^2 + y^2)$$

(5 points)

2. Conversely, show that a point $(x, y, z) \in \mathbb{R}^3$ that satisfies this equation is on the torus, i.e., can be written in the form above for suitable parameters u and v with $0 \le u, v < 2\pi$. (20 points)

(Hint: Any point (c, d) with $c^2 + d^2 = 1$ can be written in the form $c = \cos(\varphi)$, $d = \sin(\varphi)$ for a suitable parameter φ with $0 \le \varphi < 2\pi$.)

Due date: Thursday, February 18, 2021. Write your solution on letter-sized paper, scan it and submit it to the assignment box for Sheet 4 in the Brightspace site for this course. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.