Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 4230: Sheet 3**

Differential Geometry

Problem 1: Suppose that $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^3$ are two open sets, and that $f : U \to \mathbb{R}^3$ and $g : V \to \mathbb{R}^2$ are differentiable maps with $f(U) \subset V$. Then their composition $h := g \circ f$ is a map from U to $\mathbb{R}^2$. Suppose that f maps $p \in U$ to $q \in V$. State the chain rule in this situation, i.e., express the differential dh_p, which is a 2×2-matrix, in terms of the differential df_p, which is a 3×2-matrix and the differential dg_q, which is a 2×3-matrix. Write out these matrices, which are also known as the Jacobi matrices or Jacobians, explicitly. (25 points)

(Hint: The topic is discussed in detail in the appendix on continuity and differentiability of the textbook. You will find it very helpful to give it a look.)

Problem 2: In the situation of Problem 1, suppose that $U = \mathbb{R}^2$ and $V = \mathbb{R}^3$. Suppose furthermore that

$$f(u, v) = (u^2, uv, u + v)\text{and} \qquad g(x, y, z) = (e^x \cos(y), e^z \sin(y))$$

1. Compute $h := g \circ f$. (5 points)
2. For an arbitrary point $p = (u, v)$, find the differentials df_p, dg_q, and dh_p. (15 points)
3. Verify that the chain rule that you stated in Problem 1 holds in this case. (5 points)

Problem 3: In Section 2-6 on page 105 of the textbook, you find the formulas

$$\bar{\mathbf{x}}_{\bar{u}} = \mathbf{x}_u \frac{\partial u}{\partial \bar{u}} + \mathbf{x}_v \frac{\partial v}{\partial \bar{u}} \qquad \text{and} \qquad \bar{\mathbf{x}}_{\bar{v}} = \mathbf{x}_u \frac{\partial u}{\partial \bar{v}} + \mathbf{x}_v \frac{\partial v}{\partial \bar{v}}$$

1. Explain the notation and the hypotheses under which these formulas hold. (5 points)
2. Prove the formulas. (20 points)

Problem 4: In Section 2-6 on page 107 of the textbook, you find the formula

$$\bar{\mathbf{x}}_{\bar{u}} \times \bar{\mathbf{x}}_{\bar{v}} = (\mathbf{x}_u \times \mathbf{x}_v)\frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})}$$

where the cross product is denoted by the symbol $\times$ instead of $\wedge$.

1. Explain the notation and the hypotheses under which this formula holds. (5 points)
2. Prove the formula. (20 points)

(Remark: Note that this formula is already used in Paragraph 2-5 on page 100.)

Due date: Tuesday, February 9, 2021. Write your solution on letter-sized paper, scan it and submit it to the assignment box for Sheet 3 in the Brightspace site for this course. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.