Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 4230: Sheet 2**

Differential Geometry

Problem 1: Recall that a map $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined on an open set U is continuous if, for every $p \in U$ and every $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that $|f(q) - f(p)| < \varepsilon$ whenever $q \in U$ satisfies $|q - p| < \delta$. If $V \subset \mathbb{R}^2$ is a second open set, show that the preimage

$$f^{-1}(V) := \{q \in U | f(q) \in V\}$$

is open. (25 points)

(Remark: Recall that $U \subset \mathbb{R}^2$ is open if, for every $p \in U$, there is a number $\varepsilon > 0$ such that $q \in U$ whenever $q \in \mathbb{R}^2$ satisfies $|q - p| < \varepsilon$. The above property in fact characterises continuous mappings in the sense that the converse of the statement in the problem is also true. Everything said here holds in the same way for mappings $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$.)

Problem 2: Consider the plane curve $\alpha : \mathbb{R}_+ := (0, \infty) \to \mathbb{R}^2, t \mapsto (x(t), y(t))$, where

$$x(t) := \begin{cases} \cos(t) & : t < 2\pi \\ 1 & : t \geq 2\pi \end{cases} \qquad \text{and} \qquad y(t) := \begin{cases} \sin(t) & : t < 2\pi \\ t - 2\pi & : t \geq 2\pi \end{cases}$$

1. Show that α is continuously differentiable. (2 points)
2. Show that α is not twice differentiable. (2 points)
3. Sketch the curve α. (4 points)
4. Show that α is injective. (4 points)
5. Show that the inverse map $\alpha(\mathbb{R}_+) \to \mathbb{R}_+,\ \alpha(t) \mapsto t$ is not continuous. (13 points)

Problem 3: With the notation introduced in Problem 2, define on the open set $U := \mathbb{R}_+ \times \mathbb{R}$ the parametrised surface

$$f : U \to \mathbb{R}^3, (u, v) \mapsto (x(u), y(u), v)$$

1. Show that f is continuously differentiable. (2 points)
2. Show that f is not twice differentiable. (2 points)
3. Show that f is injective. (4 points)
4. Show that the differential df_p is injective for every point $p = (u, v) \in U$. (4 points)
5. Show that the inverse map $f(U) \to U,\ f(p) \mapsto p$ is not continuous. (13 points)

(Remark: Up to the fact that f is not arbitrarily often differentiable, this means that Conditions 1 and 3 in Definition 1 on page 54 of the textbook are satisfied, but Condition 2 there is not satisfied. With more effort, one can construct a very similar example which is arbitrarily often differentiable.)

Problem 4: Suppose that $f : U \to \mathbb{R}^3$ is defined on the open set $U \subset \mathbb{R}^2$ and satisfies Conditions 1, 2, and 3 in Definition 1 on page 54 of the textbook, i.e., satisfies

1. f is arbitrarily often differentiable.
2. f is injective.
3. The inverse map $f(U) \to U,\ f(p) \mapsto p$ is continuous.

   (In the spirit of Problem 1, this can also be expressed as follows: For every open set $U' \subset U$, the image $f(U')$ is open in $f(U)$, i.e., there is an open set $V' \subset \mathbb{R}^3$ with $f(U') = f(U) \cap V'$.)
4. The differential df_p is injective for every point $p = (u, v) \in U$.

Show that, for every point $p \in U$, there is an open set $W \subset \mathbb{R}^3$ with $f(p) \in W$ and a differentiable function $G : W \to U$ so that $G(f(q)) = q$ for all $q \in f^{-1}(W)$. In other words, the inverse map $f(U) \to U,\ f(q) \mapsto q$ is not only continuous, but locally even the restriction of a differentiable function G that is defined on an open set W of $\mathbb{R}^3$. (25 points)

(Hints: Proceed as in the lecture and as in the proof of Proposition 1 in Paragraph 2-3 on page 74 of the textbook:

1. Explain why you can assume that $\frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0$.
2. Define $F : U \times \mathbb{R} \to \mathbb{R}^3,\ (u, v, t) \mapsto f(u, v) + (0, 0, t)$.
3. Apply the inverse function theorem on page 133 of the textbook to find an open set $V \subset U \times \mathbb{R}$ with $(p, 0) \in V$ such that $F : V \to F(V)$ is a diffeomorphism with the open set $F(V)$.
4. Explain why $U' := \{q \in U | (q, 0) \in V\}$ is open.
5. Using Condition 2 in the textbook, i.e., Hypothesis 3 of this problem, conclude that $f(U')$ is open in $f(U)$, so that there is an open set $W' \subset \mathbb{R}^3$ with $f(U') = f(U) \cap W'$.
6. Set $W := W' \cap F(V)$. Show that $f(U) \cap W' = f(U') \subset F(V)$ and that $f^{-1}(W) = U'$.
7. Show that for G, you can use the first two components of the restriction of the inverse map $F^{-1} : F(V) \to V$ to W.)

Due date: Tuesday, February 2, 2021. Write your solution on letter-sized paper, scan it and submit it to the assignment box for Sheet 2 in the Brightspace site for this course. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.