Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 4230: Sheet 1**

Differential Geometry

Problem 1: Consider the space curve $\alpha(t) := (x(t), y(t), z(t))$, where

$$x(t) := \cos^2(t) - \frac{1}{2} \qquad y(t) := \sin(t)\cos(t) \qquad z(t) := \sin(t)$$

1. Show that $x(t)^2 + y(t)^2 = \frac{1}{4}$, and explain in words what this equation means geometrically. (8 points)
2. Show that $(x(t) + \frac{1}{2})^2 + y(t)^2 + z(t)^2 = 1$, and explain in words what this equation means geometrically. (8 points)
3. Draw a three-dimensional picture of the curve. (9 points)

Problem 2: Suppose that $\varphi : I \to \mathbb{R}$ is an arbitrarily often differentiable function defined on the interval I, and define the plane curve $\alpha(t) := (x(t), y(t))$, where

$$x(t) := \int \cos(\varphi(t))dt \qquad y(t) := \int \sin(\varphi(t))dt$$

1. If $\varphi'(t) > 0$ for all $t \in I$, show that the curvature $\kappa(t)$ of this curve is $\kappa(t) = \varphi'(t)$. (23 points)
2. Conclude that any strictly positive, arbitrarily often differentiable function defined on an interval I is the curvature of a suitable plane curve. (2 points)

Problem 3: For the torsion $\tau(t)$ of a regular space curve $\alpha : I \to \mathbb{R}^3$ defined on the interval I, prove the formula:

$$\tau(t) = -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \times \alpha''(t)|^2}$$

(25 points)

Problem 4: Suppose that $\alpha : I \to \mathbb{R}^3$ is a regular space curve defined on the interval I, and that $M \in SO(3)$ is an orthogonal matrix with determinant 1. Define the curve

$$\beta(t) := M\alpha(t) + v$$

where $v \in \mathbb{R}^3$ is a fixed vector. Show that the curves α and β have the same curvature and the same torsion. (25 points)

Due date: Tuesday, January 26, 2021. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.