Complex Numbers

A supplement for
Math 2000 and Math 2050

by

Shannon Patrick Sullivan

Department of Mathematics and Statistics
Memorial University of Newfoundland

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1 Introduction

In mathematics, we use various number systems.

The **integers** $\mathbb{Z}$ consist of the positive and negative whole numbers, as well as zero: $0, \pm 1, \pm 2, \ldots$. They suffice to solve simple equations such as

$$x - 3 = 0.$$

The **rational numbers** $\mathbb{Q}$ are quotients of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. They are needed to solve equations such as

$$4x - 5 = 0.$$

Rational numbers have a decimal representation which either terminates or includes a repeating pattern after the decimal point.

The **irrational numbers** (which have no generally accepted symbol) are solutions to equations such as

$$x^2 - 2 = 0.$$

They include any number whose decimal representation does not terminate or repeat.

Together, the rational and irrational numbers make up the **real numbers** $\mathbb{R}$.

However, even the real numbers are insufficient to solve equations such as

$$x^2 + 1 = 0. \tag{1.1}$$

The solution to Equation 1.1 would require $x = \pm\sqrt{-1}$, and we know that the square root is undefined for negative real numbers.

However, we can expand our number system beyond $\mathbb{R}$ if we define a new number i which has the property that

$$i^2 = -1.$$

This would mean that the solutions to Equation (1.1) are $x = i$ and $x = -i$.

In this manner, we can define the **complex numbers** $\mathbb{C}$ to consist of all numbers of the form $z = \alpha + i\beta$ where α and β are real numbers. We call α the **real part** of z because, if $\beta = 0$, then $z = \alpha$ is just a real number. We call β the **imaginary part** of z and, if $\alpha = 0$, we refer to z as a (pure) **imaginary number**.

Note that, in some disciplines, the symbol j is used in place of i.

Example 1.1. Solve the equation $x^2 + 4x + 5 = 0$.

Solution. Using the quadratic formula, we have

$$x = \frac{-4 \pm \sqrt{(-4)^2 - 4 \cdot 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2\sqrt{-1}}{2} = -2 \pm i. \qquad \blacksquare$$

Observe that the integers are a subset of the rational numbers; the rational and irrational numbers are both subsets of the real numbers; and the real numbers are a subset of the complex numbers.

2 Arithmetic of Complex Numbers

Complex numbers can be added or subtracted by adding or subtracting the real parts and the imaginary parts.

Example 2.1. $(1 + 2i) + (3 + 4i) = (1 + 3) + (2 + 4)i = 4 + 6i$

Complex numbers can be multiplied much like first-degree polynomials, with the added stipulation that $i^2 = -1$.

Example 2.2. $(1 + 2i)(3 + 4i) = 3 + 4i + 6i + 8i^2 = 3 + 10i + 8(-1) = -5 + 10i$

Two complex numbers are **equal** if and only if both their real parts are equal and their imaginary parts are equal. That is, if $z = \alpha + i\beta$ and $w = \gamma + i\delta$ then $z = w$ if and only if both $\alpha = \gamma$ and $\beta = \delta$.

This means that if $\alpha + i\beta = 0$ then $\alpha = \beta = 0$ because $0 = 0 + 0i$.

Some expressions which cannot be factored in terms of real numbers can be factored using complex numbers. For instance, the sum of squares can be

$$\alpha^2 + \beta^2 = (\alpha - i\beta)(\alpha + i\beta),$$

as can be shown simply by expanding the expression on the righthand side.

Example 2.3. $x^2 + 9 = (x - 3i)(x + 3i)$

The notion of the reciprocal of a complex number (and, therefore, the division of complex numbers) is a bit more involved than addition or multiplication. If $z = \alpha + i\beta \neq 0$ then

$$\begin{aligned} \frac{1}{z} &= \frac{1}{\alpha + i\beta} \\ &= \frac{1}{\alpha + i\beta} \cdot \frac{\alpha - i\beta}{\alpha - i\beta} \\ &= \frac{\alpha - i\beta}{\alpha^2 + \beta^2} \\ &= \frac{\alpha}{\alpha^2 + \beta^2} - i\frac{\beta}{\alpha^2 + \beta^2}. \end{aligned} \tag{1.2}$$

In other words, the reciprocal of z is the complex number with real part $\frac{\alpha}{\alpha^2 + \beta^2}$ and imaginary part $-\frac{\beta}{\alpha^2 + \beta^2}$. On its own, Equation (1.2) isn't particularly useful. But notice that if $z \neq 0$ then at least one of α and β is non-zero, and so $\alpha^2 + \beta^2 \neq 0$. In other words, this result does guarantee that every non-zero complex number has an inverse.

We might also observe from the derivation of Equation (1.2) that, in the second step, it was very useful to multiply both the numerator and the denominator of the expression by $\alpha - i\beta$. This is similar to the approach we take when we rationalise an expression involving square roots — and, just like in that situation, we refer to $\alpha - i\beta$ as the **conjugate** of $\alpha + i\beta$. If z is a complex number then we denote its conjugate as $\overline{z}$.

We can use the conjugate to simplify quotients of complex numbers, and therefore to carry out division without having to make direct use of Equation (1.2).

Example 2.4. $\dfrac{2+2i}{2-3i} = \dfrac{2+2i}{2-3i} \cdot \dfrac{2+3i}{2+3i} = \dfrac{4+10i-6}{4+9} = -\dfrac{2}{13} + \dfrac{10}{13}i$

We define the absolute value or **modulus** of a complex number $z = \alpha + i\beta$ as

$$|z| = \sqrt{\alpha^2 + \beta^2}.$$

Note that $|z|$ is always a real, non-negative number.

The conjugate and the modulus obey or motivate several properties of complex numbers. If z and w are complex numbers, then we have the following:

1. $\overline{z \pm w} = \overline{z} \pm \overline{w}$

2. $\overline{zw} = \overline{z} \cdot \overline{w}$

3. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

4. $\overline{(\overline{z})} = z$

5. z is real if and only if $z = \overline{z}$

6. $z\overline{z} = |z|^2$

7. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

8. $|z| \geq 0$ for all z

9. $|z| = 0$ if and only if $z = 0$

10. $|z||w| = |zw|$

11. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

12. $|z + w| \leq |z| + |w|$

All of these results can be proved using the definitions of a complex number, its conjugate and its modulus. For example, we can prove Property #2 by assuming that $z = \alpha + i\beta$ and $w = \gamma + i\delta$. Then

$$zw = (\alpha + i\beta)(\gamma + i\delta) = \alpha\gamma + i\alpha\delta + i\beta\gamma - \beta\delta = (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma)$$

so

$$\overline{zw} = (\alpha\gamma - \beta\delta) - i(\alpha\delta + \beta\gamma).$$

On the other hand,

$$\overline{z} \cdot \overline{w} = (\alpha - i\beta)(\gamma - i\delta) = \alpha\gamma - i\alpha\delta - i\beta\gamma - \beta\delta = (\alpha\gamma - \beta\delta) - i(\alpha\delta + \beta\gamma)$$

as well. Hence $\overline{zw} = \overline{z} \cdot \overline{w}$.

3 The Complex Plane

We cannot plot complex numbers on a real number line. Instead, we plot them on the **complex plane**. Like the Cartesian plane, this consists of two perpendicular axes which intersect at the origin. The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**. We then identify the complex number $\alpha + i\beta$ with the point (α, β) in this plane. In other words, the real part α determines the horizontal coordinate while the imaginary part β determines the vertical coordinate.

Example 3.1. The complex numbers $z_1 = 3 + 2i$, $z_2 = -2i$ and $z_3 = 3$ are plotted in Figure 1.1.

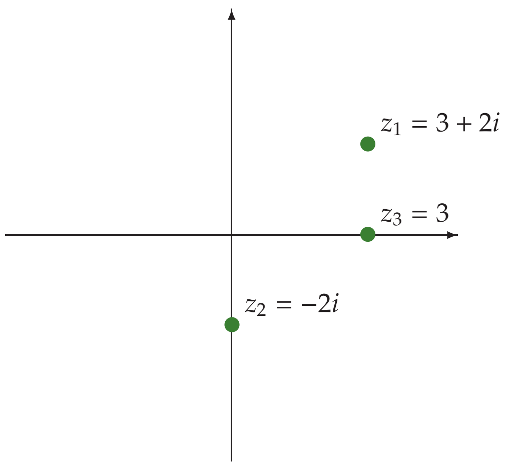

Figure 1.1: Three numbers plotted in the complex plane.

Note that real numbers are plotted on the real axis (since this is effectively just the familiar real number line) while imaginary numbers are plotted on the imaginary axis.

The distance between two numbers in the complex plane is given by the modulus of their difference. That is, if $z = \alpha + i\beta$ and $w = \gamma + i\delta$ then the distance between z and w is

$$|z - w| = \sqrt{(\alpha - \gamma)^2 + (\beta - \delta)^2}.$$

4 Math 2000: Power Series and Euler's Formula

To this point, you've probably only encountered functions where the domain and range are the real numbers $\mathbb{R}$ or a subset of $\mathbb{R}$. However, we can just as easily define functions where the domain and range are the complex numbers $\mathbb{C}$ or a subset of $\mathbb{C}$.

This is particularly easy for algebraic functions.

Example 4.1. Consider the function $f(x) = x^2$ where the domain is the set of all complex numbers $\mathbb{C}$. Determine $f(3i)$ and $f(i+2)$.

Solution. We simply evaluate

$$f(3i) = (3i)^2 = 9i^2 = -9$$

and

$$f(i+2) = (i+2)^2 = i^2 + 4i + 4 = 3 + 3i. \qquad \blacksquare$$

However, this becomes more difficult when considering transcendental functions. For instance, how can we interpret e^{3i}?

One way to obtain some insight into this topic is via power series. The Maclaurin series we've established work equally well for complex domains, so we can write

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}.$$

But now observe that $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, and then the pattern repeats *ad infinitum*. With this in mind, we can split the Maclaurin series up into the sum of two series, one for even values of k (when i^k is real) and one for odd values of k (when i^k is imaginary):

$$\begin{aligned}
\sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{(ix)^k}{k!} + \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \frac{(ix)^k}{k!} &= \sum_{\substack{k=0 \\ k=2n}}^{\infty} \frac{i^k x^k}{k!} + \sum_{\substack{k=0 \\ k=2n+1}}^{\infty} \frac{i^k x^k}{k!} \\
&= \sum_{n=0}^{\infty} \frac{i^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1} x^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

But the two series we've obtained are simply the Maclaurin series for cosine and sine, respectively. In other words,

$$e^{ix} = \cos(x) + i\sin(x). \tag{1.3}$$

Equation (1.3) is known as **Euler's formula**.

Using Euler's formula, observe that

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x).$$

This is because cosine is an even function, so $\cos(-x) = \cos(x)$ for all x, while sine is an odd function, so $\sin(-x) = -\sin(x)$ for all x.

More generally, for any real number b,

$$e^{ibx} = \cos(bx) + i\sin(bx).$$

Now we can think of any complex number $z = \alpha + i\beta$ as having an equivalent representation

$$z = \alpha + i\beta = r\cos(\theta) + ir\sin(\theta) = r[\cos(\theta) + i\sin(\theta)] = re^{i\theta},$$

where r is the **modulus** and θ is the **argument**. Thus complex numbers can be written in a polar form (r, θ), where the conversion is analogous to Cartesian coordinates. Specifically,

$$\alpha = r\cos(\theta) \quad \text{and} \quad \beta = r\sin(\theta)$$

while

$$r = \sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad \theta = \arctan\left(\frac{\beta}{\alpha}\right).$$

Example 4.2. Find the complex number z with polar coordinates $\left(2, \frac{3\pi}{4}\right)$.

Solution. Since $r = 2$ and $\theta = \frac{3\pi}{4}$, we have

$$\alpha = 2\cos\left(\frac{3\pi}{4}\right) = -\sqrt{2} \quad \text{and} \quad \beta = 2\sin\left(\frac{3\pi}{4}\right) = \sqrt{2}.$$

Thus $z = -\sqrt{2} + \sqrt{2}i$. ■

Example 4.3. Express $z = 3 - 3i$ in polar coordinates.

Solution. We have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2} \quad \text{and} \quad \theta = \arctan\left(\frac{-3}{3}\right) = -\frac{\pi}{4}$$

so the polar coordinates of z are $\left(3\sqrt{2}, -\frac{\pi}{4}\right)$. The relationship between z and its polar form is illustrated in Figure 1.2. ■

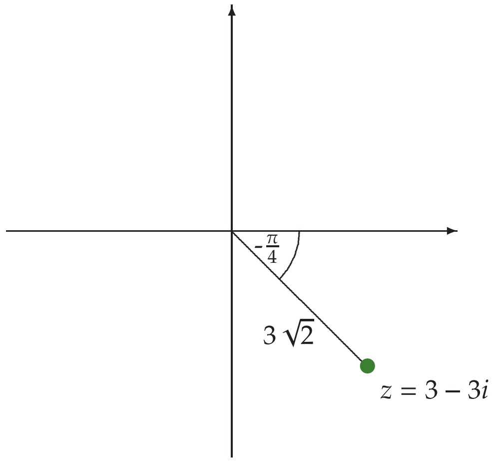

Figure 1.2: The complex number $z = 3 - 3i$ related to its modulus $r = 3\sqrt{2}$ and its argument $\theta = -\frac{\pi}{4}$.

Because it provides a way to define complex numbers without resorting to a sum, Euler's formula can make some tasks easier. For example, we can use the laws of exponents for multiplying, rather than the cumbersome expansion we encountered in Example 2.2. To see why this should be, consider two complex numbers

$$z_1 = r_1 e^{i\theta_1} = r_1[\cos(\theta_1) + i\sin(\theta_1)] \quad \text{and} \quad z_2 = r_2 e^{i\theta_2} = r_2[\cos(\theta_2) + i\sin(\theta_2)].$$

Furthermore, recall the trigonometric identities

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

and

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Then

$$\begin{aligned}
z_1 z_2 &= r_1[\cos(\theta_1) + i\sin(\theta_1)] \cdot r_2[\cos(\theta_2) + i\sin(\theta_2)] \\
&= r_1 r_2[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\cos(\theta_1)\sin(\theta_2) \\
&\quad + i\sin(\theta_1)\cos(\theta_2)] \\
&= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \\
&= r_1 r_2 e^{i(\theta_1+\theta_2)} \\
&= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}.
\end{aligned}$$

Example 4.4. Multiply $(-\sqrt{2}+\sqrt{2}i)(3-3i)$.

Solution. We have already shown that $-\sqrt{2}+\sqrt{2}i = 2e^{i\frac{3\pi}{4}}$ and $3-3i = 3\sqrt{2}e^{-i\frac{\pi}{4}}$. So we could write

$$\begin{aligned}
(-\sqrt{2}+\sqrt{2}i)(3-3i) &= 2e^{i\frac{3\pi}{4}} \cdot 3\sqrt{2}e^{-i\frac{\pi}{4}} \\
&= 6\sqrt{2}e^{i\frac{\pi}{2}} \\
&= 6\sqrt{2}i.
\end{aligned}$$

Alternatively, we could use the approach of Example 2.2 and expand:

$$\begin{aligned}
(-\sqrt{2}+\sqrt{2}i)(3-3i) &= -3\sqrt{2} + 3\sqrt{2} + 3\sqrt{2}i + 3\sqrt{2}i \\
&= 6\sqrt{2}i.
\end{aligned}$$

■

In this example, the use of Euler's formula may seem to be of negligible benefit — indeed, if we hadn't already found the polar form of the two complex numbers, it would really be more work than our earlier approach. But the utility of Euler's formula can be seen in more substantial calculations.

Example 4.5. Evaluate $(3-3i)^6$.

Solution. In this kind of calculation, multiplying by expanding becomes extremely time-consuming. But, if we again recall that $3-3i = 3\sqrt{2}e^{-i\frac{\pi}{4}}$, then we can use Euler's formula to simply write

$$\begin{aligned}(3-3i)^6 &= \left(3\sqrt{2}e^{-i\frac{\pi}{4}}\right)^6 \\ &= 3^6 \cdot 2^3 \cdot e^{-i\frac{3\pi}{2}} \\ &= 5832e^{-i\frac{3\pi}{2}} \\ &= 5832i. \qquad \blacksquare\end{aligned}$$

The result that

$$(e^{i\theta})^n = e^{in\theta} \tag{1.4}$$

is known as **DeMoivre's Theorem**.

5 Math 2050: Complex Numbers and Vectors

The components of a vector can just as easily be complex numbers as real numbers. In this case, the space of vectors with n complex components is denoted by $\mathbb{C}^n$. Similarly, scalars can be generalised to include complex numbers as well as real numbers.

Most of the properties we have defined for real vectors apply equally to complex vectors. However, the definition of the dot product as the sum of the products of the components poses a problem. Recall that, for a real vector $\mathbf{v}$, we have $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$. If the same definition of the dot product applied to complex vectors then we could consider $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and write

$$\|\mathbf{v}\| = \sqrt{i^2 + 1^2} = \sqrt{-1+1} = \sqrt{0} = 0.$$

Unfortunately, this contradicts our instinct that any non-zero vector should have a non-zero length. At the same time, we don't want to redefine the dot product for complex vectors in a way that's inconsistent with real vectors.

With this in mind, the **complex dot product** of vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbf{C}^n$ is defined to be

$$\mathbf{v}\cdot\mathbf{w} = \overline{v_1}w_1 + \cdots + \overline{v_n}w_n.$$

If the components of the vectors are real then this reduces back to our original definition of the dot product, since a real number is its own complex conjugate. Now if $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ then

$$\|\mathbf{v}\| = \sqrt{\bar{i} \cdot i + \bar{1} \cdot 1} = \sqrt{-i \cdot i + 1 \cdot 1} = \sqrt{1+1} = \sqrt{2}.$$

We can also return to the way we plotted complex numbers in the complex plane. Observe that a complex number $z = \alpha + i\beta$ can be thought of in the same terms as a vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ in $\mathbb{R}^2$. Thus, another way to justify many of the properties of complex numbers is to think of them in terms of vectors. For instance, the process of complex addition can be interpreted as an analogue of the Parallelogram Rule for vector addition; see Figure 1.3.

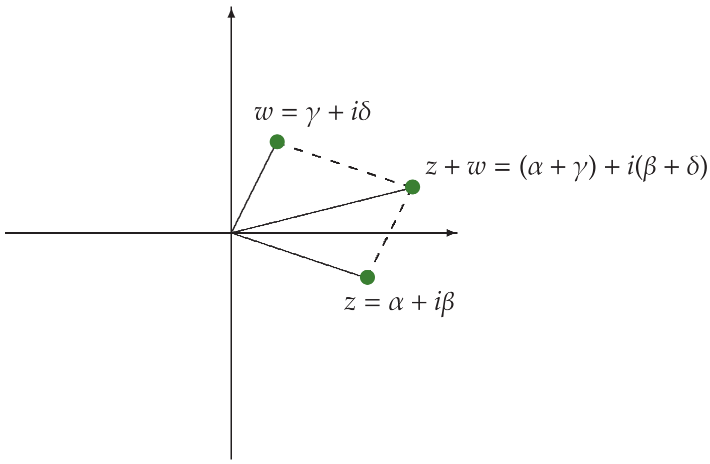

Figure 1.3: Complex addition interpreted via the Parallelogram Rule.