Memorial University of Newfoundland **Winter Semester 2024**
Yorck Sommerhäuser **MATH 3210: Sheet 8**

Introduction to Complex Analysis

Problem 1: Explain how the integral

$$\int_1^i z^5 dz$$

is defined using contour integrals and find its value. (25 points)

Problem 2: Let C be the ellipse given by the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

oriented positively, i.e., counterclockwise. Find the contour integral

$$\int_C \frac{1}{z} dz$$

(25 points)

Problem 3: Let $D \subset \mathbb{C}$ be a domain in the Argand plane, and $f : D \to \mathbb{C}$ be a continuous function. If $\gamma : [c, d] \to D \cap \mathbb{R}$ is a contour that does not leave the real line, show that

$$\int_\gamma f(z)dz = \int_a^b f(x)dx$$

where $a = \gamma(c)$ and $b = \gamma(d)$. Here, the left-hand side is a contour integral as defined in Section 39 of the textbook, while the right-hand side is an integral of a complex-valued function of a real variable as defined in Section 37 of the textbook, namely the restriction of f to the real line. This shows that contour integrals generalize the Riemann integrals from Calculus II. (25 points)

Problem 4: The Cauchy integral formula for the n-th derivative states that, if C is a positively oriented simple closed contour and f is analytic on C and inside C, then the n-th derivative of f exists at all points z inside C, and is given by the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}} ds$$

Note that the case $n = 0$ is the ordinary Cauchy integral formula for $f = f^{(0)}$. Prove this formula in the case $n = 1$.

(Hint: The proof can be found in Section 48 of the textbook; you are only supposed to reproduce it.) (25 points)

Due date: Monday, March 25, 2024. Please write your solution on letter-sized paper, and write your name on your solution. Please give all computations in full detail, and explain your computations in English, using complete sentences. It is not necessary to copy down the problems again or to submit this sheet with your solution.