Memorial University of Newfoundland **Winter Semester 2024**
Yorck Sommerhäuser **MATH 3210: Sheet 5**

Introduction to Complex Analysis

Problem 1: Use the Cauchy-Riemann equations in polar form (Sec. 22, Eq. (6), p. 66) to decide where the functions

1. $f(z) = \sqrt{r}e^{2\theta i}$ (for $r > 0$)
2. $g(z) = e^{-\theta}\cos(\ln(r)) + ie^{-\theta}\sin(\ln(r))$ (for $r > 0$)

are complex differentiable. Here we have $z = re^{\theta i}$. If $f(z)$ or $g(z)$ are differentiable, find the derivative with the help of the formula for the derivative in polar coordinates (Sec. 22, Eq. (7), p. 67). (12 points each)

Problem 2: Decide whether the function

$$h(x, y) = xy - x + y$$

is harmonic. If it is harmonic, find a harmonic conjugate. (20 points)

Problem 3: The Wirtinger derivatives defined in Problem 4 on Sheet 4 are also denoted by

$$f_z(z) = \frac{\partial f}{\partial z}(z) \qquad f_{\bar{z}}(z) = \frac{\partial f}{\partial \bar{z}}(z)$$

Show that

$$\frac{\partial z}{\partial z}(z) = 1 \qquad \frac{\partial z}{\partial \bar{z}}(z) = 0 \qquad \frac{\partial \bar{z}}{\partial z}(z) = 0 \qquad \frac{\partial \bar{z}}{\partial \bar{z}}(z) = 1$$

(20 points)

Problem 4:

1. In a calculus textbook of your choice, look up what the chain rule for a real-valued function of two real variables for the outer function states. State it in a long form which makes clear at which points the outer and the inner derivatives are evaluated. Cite the calculus textbook that you use. (6 points)

2. Suppose that $f : D \to \mathbb{C}$ and $g : E \to \mathbb{C}$ are real differentiable functions defined in domains D and E in the complex plane. If $f(D) \subset E$, deduce from the first part that we have

$$(g \circ f)_z(z) = g_w(f(z))f_z(z) + g_{\bar{w}}(f(z))\bar{f}_z(z)$$

for the Wirtinger derivatives defined in Problem 4 on Sheet 4, where the complex conjugate of a function is defined as $\bar{f}(z) := \overline{f(z)}$. (20 points)

3. Similarly, show that $(g \circ f)_{\bar{z}}(z) = g_w(f(z))f_{\bar{z}}(z) + g_{\bar{w}}(f(z))\bar{f}_{\bar{z}}(z)$. (10 points)

Due date: Monday, March 4, 2024. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.