Memorial University of Newfoundland **Winter Semester 2024**
Yorck Sommerhäuser **MATH 3210: Sheet 4**

Introduction to Complex Analysis

Problem 1: From the definition, show that $f(z) = \frac{1}{z}$ has the (complex) derivative $f'(z) = -\frac{1}{z^2}$. (25 points)

Problem 2: For the function

$$f(z) := \begin{cases} \frac{\bar{z}^2}{z} & : z \neq 0 \\ 0 & : z = 0 \end{cases}$$

find all the points at which the Cauchy-Riemann equations are satisfied.
(25 points)

Problem 3: For the function considered in Problem 2, decide whether $f'(0)$ exists. (25 points)

Problem 4: Suppose that $D \subset \mathbb{C}$ is a domain and that the function $f : D \to \mathbb{C}$ is differentiable in the real sense, i.e., as a function of two real variables. As usual, we use the notation $f(z) = u(z) + iv(z)$ for the real and the imaginary part of f.

Define the Wirtinger derivatives of f at a point $z_0 \in D$ as

$$f_z(z_0) := \frac{1}{2}(f_x(z_0) - if_y(z_0)) \qquad f_{\bar{z}}(z_0) := \frac{1}{2}(f_x(z_0) + if_y(z_0))$$

where $f_x(z_0) = u_x(z_0) + iv_x(z_0)$ and $f_y(z_0) = u_y(z_0) + iv_y(z_0)$ are the partial derivatives in the sense of real analysis. Show that f is complex differentiable at z_0 if and only if $f_{\bar{z}}(z_0) = 0$. In this case, show that $f_z(z_0) = f'(z_0)$, i.e., that the other Wirtinger derivative coincides with the derivative introduced in Section 18 of the textbook. (25 points)

Due date: Monday, February 12, 2024. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.