Introduction to Complex Analysis

Problem 1: Explain how the integral

$$\int_1^i z^5 dz$$

is defined using contour integrals and find its value. (25 points)

Problem 2: Without computing the integral explicitly, use the standard estimate from Section 41 of the textbook to show that the contour integral

$$\int_C \frac{z^3 + z + 2}{z^7 + z^3 + 1} dz$$

where C is a quarter-circle of radius 2 around the origin in the first quadrant, cannot be larger than $12\pi/119$. (25 points)

Problem 3: Let C be the ellipse given by the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

oriented positively, i.e., counterclockwise. Find the contour integral

$$\int_C \frac{1}{z} dz$$

(25 points)

Problem 4: Let $D \subset \mathbb{C}$ be a domain in the Argand plane, and $f : D \to \mathbb{C}$ be a continuous function. If $\gamma : [c, d] \to D \cap \mathbb{R}$ is a contour that does not leave the real line, show that

$$\int_\gamma f(z) dz = \int_a^b f(x) dx$$

where $a = \gamma(c)$ and $b = \gamma(d)$. Here, the left-hand side is a contour integral as defined in Section 39 of the textbook, while the right-hand side is an integral of a complex-valued function of a real variable as defined in Section 37 of the textbook, namely the restriction of f to the real line. This shows that contour integrals generalize the Riemann integrals from Calculus II. (25 points)

Due date: Tuesday, November 6, 2018. Please write your solution on letter-sized paper, and write your name on your solution. Please give all computations in full detail, and explain your computations in English, using complete sentences. It is not necessary to copy down the problems again or to submit this sheet with your solution.