Memorial University of Newfoundland **Fall Semester 2018**
Yorck Sommerhäuser **MATH 3210: Sheet 10**

Introduction to Complex Analysis

Problem 1: Consider the sequence $(f_n)_{n\in\mathbb{N}}$ that is formed by the functions $f_n(z) := z^n$.

1. Decide whether the sequence converges on the set $D := \{z \in \mathbb{C} \mid |z| < \frac{1}{2}\}$. If it converges, find its limit, i.e., the function to which it converges.
2. Decide whether the sequence converges uniformly on the set D.
3. Decide whether the sequence converges on the set $E := \{z \in \mathbb{C} \mid |z| < 1\}$. If it converges, find its limit, i.e., the function to which it converges.
4. Decide whether the sequence converges uniformly on the set E.

Problem 2: Find the Maclaurin series for the function

$$f(z) = \frac{1}{(z+1)^3}$$

in closed form, and determine its radius of convergence.
(Hint: Recall that, by definition, a Maclaurin series is a Taylor series about the origin, i.e., with $z_0 = 0$. Use differentiation of power series, starting from a variant of the geometric series.)

Problem 3: Find the Maclaurin series for the function

$$f(z) = \mathrm{Log}(1+z)$$

in closed form, and determine its radius of convergence.
(Hint: Start from the Maclaurin series of the derivative. The symbol Log denotes the principal branch of the logarithm function.)

Problem 4: For the function

$$f(z) = \frac{1}{z^2 - 2z - 24}$$

find the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-6)^n$$

about the point 6 that converges for points close to 6. Make a table with the explicit values c_{-3}, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3, and determine where precisely the Laurent series converges.

Problem 5: Consider the function

$$f(z) := \frac{z+1}{z^2 - 3z - 10}$$

If C is a circle of radius 6 around the origin, oriented counterclockwise, find

$$\int_C f(z)dz$$

(Hint: Use Cauchy's residue theorem.)

Problem 6: Recall the definition of the complex sine and cosine functions from Problem 1 on Sheet 8. The complex tangent, cotangent, secant, and cosecant functions are defined as in the real case

$$\tan(z) := \frac{\sin(z)}{\cos(z)} \qquad \cot(z) := \frac{\cos(z)}{\sin(z)}$$
$$\sec(z) := \frac{1}{\cos(z)} \qquad \csc(z) := \frac{1}{\sin(z)}$$

They are defined where the denominator is nonzero.

1. Show that the origin is a simple pole of the cotangent function, and determine its residue there.

2. If C is a circle of radius 3 around the origin, oriented counterclockwise, find

   $$\int_C \cot(z)dz$$

   (Hint: Recall that we derived the Maclaurin series for sine and cosine in class, and that we also saw in class that all zeroes of the complex sine and cosine function are real.)

Due date: There is no due date. The completion of this sheet is voluntary. The solutions will not be collected and will not be marked. However, these exercises provide valuable practice for the final exam, which takes place on Wednesday, December 5, from 9:00 am to 12:00 m, in our usual classroom.