Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 3320: Sheet 9**

Abstract Algebra

Problem 1: An ideal I in a commutative ring R is called a prime ideal if and only if, for all $r, s \in R$, we have that $rs \in I$ implies that $r \in I$ or $s \in I$. In the case where $R = \mathbb{Z}$ and $I = p\mathbb{Z}$ consists of the multiples of the integer p, show that I is a prime ideal if and only if p is a prime number. (25 points)

Problem 2: As on Sheet 8, consider a commutative ring R, a multiplicatively closed subset $S \subset R$, and the associated ring of quotients Q.

1. If $I \subset R$ is an ideal of R, show that
$$IQ := \left\{ \frac{r}{s} \;\middle|\; r \in I, s \in S \right\}$$
is an ideal of Q. (13 points)

2. If $J \subset Q$ is an ideal of Q, show that
$$J_R := \left\{ r \in R \;\middle|\; \frac{r}{1} \in J \right\}$$
is an ideal of R. (12 points)

Problem 3: We remain in the situation of the preceding problem.

1. If $J \subset Q$ is an ideal of Q, show that $J_R Q = J$. (10 points)

2. If $I \subset R$ is a prime ideal of R with $I \cap S = \emptyset$, show that $(IQ)_R = I$. (15 points)

Problem 4: Suppose that $c \in \mathbb{R}$ is a real number with the property that $c \notin \mathbb{Q}$, but $c^2 \in \mathbb{Q}$, and define
$$K := \{a + bc \mid a, b \in \mathbb{Q}\}$$

1. Show that K is a subring of $\mathbb{R}$. (15 points)

2. Suppose that $a, b \in \mathbb{Q}$ are rational numbers and that $a + bc \in K$ is nonzero. Show that $(a + bc)(a - bc) = a^2 - b^2c^2$ is a nonzero rational number. (5 points)

3. Show that every nonzero element of K is invertible in K, so that K is a field. (5 points)

(Remark: Very similar statements also hold if c is a complex number instead of a real number.)

Due date: Wednesday, November 22, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.