Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 3320: Sheet 8**

Abstract Algebra

Problem 1: Suppose that R is a commutative ring. A subset S of R is called multiplicatively closed if

1. $1 \in S$
2. For all $s, s' \in S$, we have $ss' \in S$.

Given a multiplicatively closed set S, we introduce an equivalence relation on the Cartesian product $R \times S$ by defining

$$(r, s) \sim (r', s') \text{ if and only if there exists } s'' \in S \text{ so that } rs's'' = r'ss''$$

We denote the equivalence class of (r, s) by $\frac{r}{s}$ or r/s.

1. Show that the relation just defined is indeed an equivalence relation; i.e., that it is reflexive, symmetric, and transitive. (15 points)
2. Show that, if $0 \in S$, we have $r/s = 0/1$ for all $r \in R$ and $s \in S$. (5 points)
3. Show that, if $0 \notin S$ and R is an integral domain, we have $(r, s) \sim (r', s')$ if and only if $rs' = r's$. (5 points)

Problem 2: In the situation of Problem 1, denote the set of equivalence classes by Q.

1. Show that Q becomes a commutative ring if addition and multiplication are defined by

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'} \qquad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$$

   State explictly what the respective unit elements 0_Q and 1_Q are. (20 points)

2. Show that the map

$$i \colon R \to Q,\ r \mapsto i(r) := \frac{r}{1}$$

   is a ring homomorphism. (5 points)

Q is called the ring of quotients or the ring of fractions of R with respect to the multiplicatively closed set S, to be distinguished from the quotient ring or factor ring of R with respect a two-sided ideal.

Problem 3: We remain in the situation of the two preceding problems.

1. Show that for all $s \in S$, the element $i(s) \in Q$ is a unit, i.e., a multiplicatively invertible element. (10 points)

2. Suppose that R is an integral domain with $1_R \neq 0_R$, and let $S = R \setminus \{0\}$ be the set of nonzero elements. Show that S is multiplicatively closed, that Q is a field, and that $i\colon R \to Q$ is injective. (15 points)

(Remark: In the case where $R = \mathbb{Z}$, the ring of integers, the field Q just constructed is the field $\mathbb{Q}$ of rational numbers.)

Problem 4: As before, suppose that R is a commutative ring and that S is a multiplicatively closed subset of R with associated ring of quotients Q. Suppose that T is a not necessarily commutative ring and that $f\colon R \to T$ is a ring homomorphism with the property that $f(s)$ is a unit for all $s \in S$.

1. For $r \in R$ and $s \in S$, show that $f(r)$ and $f(s)^{-1}$ commute, i.e., that we have $f(r)f(s)^{-1} = f(s)^{-1}f(r)$. (5 points)

2. Show that there is a unique ring homomorphism $g\colon Q \to T$ with the property that $f = g \circ i$. (20 points)

Due date: Wednesday, November 15, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.