Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 3320: Sheet 7**

Abstract Algebra

Problem 1: Suppose that R is a commutative ring. On the Cartesian product $C := R \times R$, we define two binary operations $+\colon C \times C \to C$ and $\cdot\colon C \times C \to C$ by the formulas

$$(a, b) + (c, d) := (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) := (ac - bd, ad + bc)$$

Show that C is again a commutative ring. State explicitly what the neutral elements 0_C and 1_C are. (25 points)

Problem 2: In the situation of Problem 1, define the mappings $j\colon R \to C$ and $\bar{}\colon C \to C$ by the formulas

$$j(a) := (a, 0) \quad \text{and} \quad \overline{(a, b)} := (a, -b)$$

1. Show that j is a ring homomorphism. (5 points)
2. Show that $\bar{}$ is a ring homomorphism. (10 points)
3. Show that
$$(a, b) \cdot \overline{(a, b)} = (a^2 + b^2, 0) = j(a^2 + b^2)$$
(5 points)
4. For the element $i := (0, 1) \in C$, show that $i^2 = (-1, 0) = j(-1)$. (5 points)

Problem 3: In the case where $R = \mathbb{R}$, the ring of real numbers, the ring C constructed in Problem 1 is denoted by $\mathbb{C}$ and is called the ring of complex numbers. Show that $\mathbb{C}$ is in fact a field, not only a commutative ring. (25 points)

Problem 4: Show that in the case where $R = \mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$, the ring of integers modulo 3, the ring C constructed in Problem 1 is also a field. (25 points)

Due date: Monday, November 6, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.