Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 3320: Sheet 6**

Abstract Algebra

Problem 1: In the situation of Problem 1 on Sheet 4, define the mappings

$$i_N \colon N \to G,\ n \mapsto (n, e_H) \quad \text{and} \quad i_H \colon H \to G,\ h \mapsto (e_N, h)$$

and denote their images by $\tilde{N} = i_N(N)$ and $\tilde{H} = i_H(H)$, respectively.

1. Show that i_N and i_H are group homomorphisms. (10 points)
2. Show that $\tilde{N}$ and $\tilde{H}$ are subgroups of $G = N \rtimes H$ with the property that $\tilde{N} \cap \tilde{H} = \{e_G\}$. (5 points)
3. Show that $\tilde{N}$ is a normal subgroup of G. (10 points)

Problem 2: Suppose that N and H are subgroups of the finite group G with $N \cap H = \{e_G\}$ and $|N||H| = |G|$. Suppose further that N is a normal subgroup. Define $\varphi \colon H \to \mathrm{Aut}(N)$ by setting

$$\varphi(h)(n) := hnh^{-1}$$

1. Show that φ is a group homomorphism. (5 points)
2. Show that

$$f \colon N \rtimes H \to G,\ (n, h) \mapsto nh$$

is a group isomorphism. (20 points)

(Remark: Compare with Corollary 2 in Section 2.8 of the textbook, where it is assumed in addition that also H is normal.)

Problem 3: In $G = A_4$, the alternating group on four letters, consider the Klein four group

$$N = \{\mathrm{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

Find a subgroup H of G that satisfies the hypotheses in Problem 2. (25 points)

Problem 4: In $G = Q$, the quaternion group defined in Problem 2 on Sheet 5, consider the cyclic subgroup

$$N := \langle I \rangle = \{E, I, -E, -I\}$$

generated by I. Show that there is no subgroup H of G that satisfies the hypotheses in Problem 2. (25 points)

Due date: Monday, October 30, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.