Abstract Algebra

Problem 1: Suppose that G is a finite group and that H is a subgroup of G of index $[G:H] = 2$. Show that H is a normal subgroup. (25 points)

Problem 2: Inside the set $M(2 \times 2, \mathbb{C})$ of 2×2-matrices with entries from the set of complex numbers $\mathbb{C}$, consider the elements

$$E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad K := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

1. Make a multiplication table for these matrices. (10 points)
2. Show that the set

$$Q := \{E, I, J, K, -E, -I, -J, -K\}$$

is contained in the group $\mathrm{GL}(2, \mathbb{C})$ of invertible 2×2-matrices with complex entries, and is in fact a subgroup of this group, called the quaternion group. (15 points)

(Hint: The only fact about complex numbers that you need to know here is that $i^2 = -1$.)

Problem 3: Show that every subgroup of the quaternion group is normal.

(Hint: Use Problem 1. A group with the property that each of its subgroups is normal is called a Dedekind group, or, if it is non-abelian, a Hamiltonian group.) (25 points)

Problem 4: For a group G, denote by $\mathrm{Aut}(G)$ the group of group automorphisms of G and by $\mathrm{Inn}(G)$ its subgroup of inner automorphisms. Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. (25 points)

Due date: Monday, October 23, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.