Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 3320: Sheet 4**

Abstract Algebra

Problem 1: Suppose that N and H are groups and that $\varphi\colon H \to \mathrm{Aut}(N)$ is a group homomorphism. On the Cartesian product $G := N \times H$, define the multiplication

$$(n,h)(n',h') := (n(\varphi(h)(n')), hh')$$

Show that G is a group when endowed with this multiplication, the so-called semidirect product of N and H, denoted by $N \rtimes H$.
(Hint: The inverse element of a given element is $(n,h)^{-1} = (\varphi(h^{-1})(n^{-1}), h^{-1})$. You still need to show that this is indeed the case, though.) (25 points)

Problem 2: Suppose that $N = \langle a \rangle$ is a cyclic group of order m and that $H = \langle b \rangle$ is a cyclic group of order 2, so that $H = \{e_H, b\}$. Define $\varphi\colon H \to \mathrm{Aut}(N)$ by requiring that $\varphi(e_H) = \mathrm{id}_N$ and

$$\varphi(b)(n) = n^{-1}$$

for all $n \in N$.

1. Show that $\varphi(b)$ is indeed an automorphism of N. (5 points)
2. Show that φ is a group homomorphism. (5 points)
3. In the semidirect product $G := N \rtimes H$, consider the elements $c := (a, e_H)$ and $s := (e_N, b)$. Show that they satisfy the relations

$$c^m = e_G \qquad s^2 = e_G \qquad csc = s$$

(15 points)

(Remark: The group G is called the dihedral group and is denoted by D_m in our textbook, where it is constructed differently, however.)

Problem 3: In the situation of Problem 2, define $t := (a, b)$.

1. Show that $t^2 = e_G$. (5 points)
2. Show that s and t generate $G = N \rtimes H$. (20 points)

(Hint: For the notion of the subgroup generated by a subset, see the end of Section 2.4 in the textbook.)

Problem 4: Show that the dihedral group D_3 is isomorphic to the symmetric group S_3. Give an explicit isomorphism. (25 points)

Due date: Monday, October 16, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.