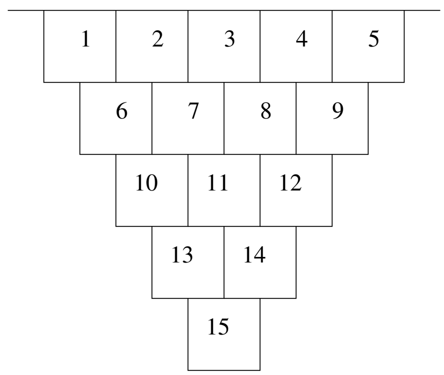
Tufts University
Department of Mathematics
Math 87 Homework 5

Due: Thursday, November 1, at 1:30 p.m. (in class).

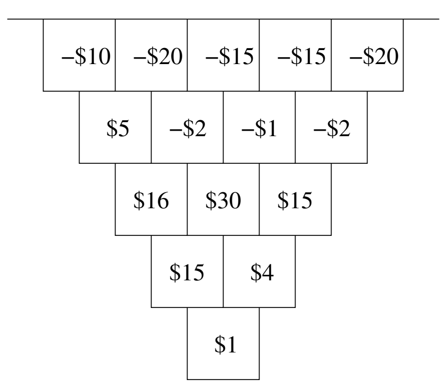
1. In this problem, we examine the question of maximizing the profit from an open pit mine from the point of view of a max-flow/min-cut formulation. Below is a schematic of an open-pit mine. In order to ensure the mine doesn't collapse, it can only be excavated in a safe way; in order to excavate a block below the first level, both blocks directly above it must also be excavated (and, if applicable, both blocks above each of these and so on). Thus, to excavate block 6, we need to excavate both blocks 1 and 2. To excavate block 12, we must excavate both blocks 8 and 9, requiring excavation of blocks 3, 4, and 5.



   (a) (20 points) From the above diagram, draw a directed graph with vertices, $\mathcal{V}$, corresponding to the blocks (keeping the numbering from 1-15), and directed edges, $\mathcal{E}$, for each direct dependence of the form $(u, v) \in \mathcal{E}$ if excavation of block u requires excavation of block v. (Thus, for example, edge $(12, 8)$ should be in the graph, but not edge (12,3), since that dependence is not *direct*.)

   (b) (5 points) A "safe" excavation corresponds to a "closed subset", $\mathcal{C}$ of $\mathcal{V}$. Subset $\mathcal{C}$ is closed if, for every $u \in \mathcal{C}$, if $(u, v) \in \mathcal{E}$, then $v \in \mathcal{C}$. Circle a closed subset in your graph from part (a).

   (c) (25 points) If we additionally endow each block with a value, given as the expected difference between profit from selling the ore extracted when excavating and the costs of excavation, we can augment the graph in part (a) in order to calculate the maximum possible profit from the mining. The figure below gives the values for this problem.



Redraw your graph from part (a), adding two vertices, s and t. Add a directed edge from s to each vertex for which the value of the corresponding block is positive, and a directed edge to t from each vertex for which the value of the corresponding block is negative. Add (positive) weights to the edges with the value of the block for an edge from s, with minus the (negative) value of the block for an edge to t, and infinite weight for an edge in the original graph from part (a). For example, there should be an edge from s to vertex 15 with weight 1, and an edge from vertex 1 to t with weight 10.

(d) (5 points) A maximum profit corresponds to minimum cut in the graph. With edges of infinite weight, however, it isn't necessarily guaranteed that there is a finite cut (i.e., a partition that splits s and t where no edge of infinite weight is cut between the two parts of the graph). Circle a finite cut in your graph from part (c).

(e) (25 points) Find the minimum cut in the graph, both its value and which edges are cut to give it. (Remember that an edge contributes to the value of the cut only if it starts in the piece connected to s and ends in the piece connected to t; thus, your cut can cross infinite-weight edges, so long as they are edges from the t-piece to the s-piece.) Nodes that stay in the piece connected to s represent those that are to be excavated, while those in the piece connected to t represent those that are not excavated.

Calculate the profit that the min cut corresponds to. Note that it isn't obvious that the min cut corresponds to a maximum profit. To see this, consider the finite-weight edges that are cut: some will come from s to nodes that aren't excavated, and some will come from nodes that are excavated to t. Adding and subtracting the sum of the weights on edges from s that aren't cut shows that the profit is given by the difference between the total of all the weights on edges from s and the min cut. (You don't need to prove that, just use it!)

2. (20 points) Draw a bipartite graph with four nodes on each side and seven edges in which a perfect matching is possible. Draw one where a perfect matching is impossible. What is the size of the maximum matching in your second example? How many edges do you need to add to your second example in order for a perfect matching to again be possible?