Homework 3

(Due Friday, March 15, 2013)

1. Verify that, for any $k \in \mathbb{Z}$, the function $u_k(x) = \sin\left((k + \frac{1}{2})\pi x\right)$ is a solution of the differential equation $-u''(x) = f(x)$ on $(0, 1)$, with boundary conditions $u(0) = 0$ and $u'(1) = 0$. What right-hand side, $f(x)$, is needed for this to hold?
2. Implement a code to compute the finite-element solution to the differential equation $-u''(x) = f(x)$ on $(0, 1)$, with boundary conditions $u(0) = 0$, $u'(1) = 0$. Your code should take, as input, an array of n points $0 = x_1 < x_2 < \ldots < x_n = 1$, and return, as output, the values of the finite-element solution at these points, $u(x_i)$, $i = 1, \ldots, n$.
3. Verify that the error in your finite-element solution satisfies the expected bound, depending on h and $\|f\|$. Remember that these bounds are in terms of norms of the functions and not vector norms of your discrete solution. (How are the vector norms and function norms related?) In particular, show that the error decreases by an appropriate factor when h is repeatedly halved.
4. Consider solutions, $u(x)$, of the differential equation that have different amounts of smoothness ($u \in \mathcal{C}^\ell$ for different ℓ). How does the dependence of the error in the finite-element solution change with smoothness of $u(x)$ (or $f(x)$)? You may find it easier to modify your code from Problem 2 to allow for Dirichlet BCs at both ends to investigate this.