Tufts University
Department of Mathematics
Math 250-03 Take-Home Midterm Exam

Due: Thursday, October 25, at 3:00 p.m. (in class). No extensions!

1. (15 points) Let $K(x)$ be a bounded positive function for $x \in [0,1]$. Solve

$$\partial_x \left(K(x)\partial_x u(x)\right) = 0 \text{ for } 0 < x < 1,$$

with $u(0) = 0$ and $u(1) = 1$.

2. (25 points) Consider Poisson's Equation on the unit square in $\mathbb{R}^2$ with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\partial_{xx}u - \partial_{yy}u = 0 & 0 < x < 1, 0 < y < 1 \\ u(x,0) = 1 - x & 0 \le x \le 1 \\ u(x,1) = 0 & 0 \le x \le 1 \\ u(0,y) = 1 - y & 0 \le y \le 1 \\ u(1,y) = 0 & 0 \le y \le 1 \end{cases}.$$

Using separation of variables, find a formal series solution for $u(x,y)$. *Hint:* It is much easier to solve this problem if you write $u(x,y) = u_1(x,y) + u_2(x,y)$, where $u_1(x,y)$ satisfies

$$\begin{cases} -\partial_{xx}u_1 - \partial_{yy}u_1 = 0 & 0 < x < 1, 0 < y < 1 \\ u_1(x,0) = 1 - x & 0 \le x \le 1 \\ u_1(x,1) = 0 & 0 \le x \le 1 \\ u_1(0,y) = 0 & 0 \le y \le 1 \\ u_1(1,y) = 0 & 0 \le y \le 1 \end{cases},$$

and $u_2(x,y)$ satisfies

$$\begin{cases} -\partial_{xx}u_2 - \partial_{yy}u_2 = 0 & 0 < x < 1, 0 < y < 1 \\ u_2(x,0) = 0 & 0 \le x \le 1 \\ u_2(x,1) = 0 & 0 \le x \le 1 \\ u_2(0,y) = 1 - y & 0 \le y \le 1 \\ u_2(1,y) = 0 & 0 \le y \le 1 \end{cases}.$$

Begin by explaining why this is possible.

3. (25 points) The wave equation derived in class models a string with no internal stiffness. When stiffness is important, the appropriate PDE is

$$\partial_{tt}u = c^2\partial_{xx}u - a^2\partial_{xxxx}u \text{ for } 0 < x < L,$$

with boundary and initial conditions

$$\begin{array}{ll} u(0,t) = u(L,t) = 0 & t \ge 0 \\ u_{xx}(0,t) = u_{xx}(L,t) = 0 & t \ge 0 \\ u(x,0) = f(x) & 0 \le x \le L \\ u_t(x,0) = g(x) & 0 \le x \le L. \end{array}$$

Using an energy argument, prove uniqueness of solutions to these equations. State explicitly your assumptions on the solutions.

4. (15 points) Consider the wave equation with initial and final values, instead of two initial conditions:

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < 1, 0 < t < 2 \\ u(0,t) = 0 & 0 \le t \le 2 \\ u(1,t) = 0 & 0 \le t \le 2 \\ u(x,0) = f(x) & 0 \le x \le 1 \\ u(x,2) = g(x) & 0 \le x \le 1 \end{cases}.$$

Show that the wave equation with this combination of initial and final values does not have a unique solution, by choosing $f(x)$ and $g(x)$ such that two solutions exist. Be sure to explain how you choose $f(x)$ and $g(x)$.

5. (10 points) Consider the sequence of functions given by

$$\phi_n(x) = \begin{cases} 2^{n/2} & 2^{-n} < x < 2^{1-n} \\ 0 & \text{otherwise} \end{cases}.$$

Show that $\{\phi_n\}_{n\ge1}$ is an orthonormal sequence in $L_2[0,1]$. Does it form a basis?

6. (10 points) Let $f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$ and $g(x) = \sum_{n=1}^{\infty} d_n \sin(n\pi x)$ be functions in $L_2[0,1]$. Compute $\int_0^1 f(x)g(x)dx$.