Tufts University
Department of Mathematics
Math 250-03 Homework 8

Due: Thursday, November 29, at 3:00 p.m. (in class).

1. (15 points) Let $\delta(x)$ be the Dirac delta distribution, and $\delta^{(m)}(x)$ be its m^{th} derivative, as in Definition 1.26. Show that

$$x^n \delta^{(m)}(x) = \begin{cases} 0 & \text{if } m < n \\ (-1)^n n! \delta(x) & \text{if } m = n \\ \frac{(-1)^n m!}{(m-n)!} \delta^{(m-n)}(x) & \text{if } m > n \end{cases}$$

where the equality is interpreted in terms of being equal as distributions. Hint: the "general Leibniz rule" is both good to know and helpful here.

2. (20 points) Find the fundamental solution for the operator $L = D^2 - 4D + 4$.

3. (20 points) Find the Green's function for $L = D^2 - 4$ with boundary conditions $u(0) = 0$, $u'(1) = 0$.

4. (15 points) In a general metric space, M, a set $A \subset M$ is compact if and only if it is complete and totally bounded. Theorem 6.33 in Griffel essentially proves this, where A is totally bounded when $\forall \epsilon > 0$ there is a finite collection of open disks,

$$D(x_i, \epsilon) = \{y \in M \mid d(y, x_i) < \epsilon\},$$

whose union covers A. If M is a normed space, $d(y, x_i) = \|x - y_i\|$. Corollary 6.34 proves that bounded sets in $\mathbb{R}^n$ are totally bounded. Extend this proof to show that bounded sets in any finite-dimensional inner product space are totally bounded. (The reason for considering only inner product spaces is twofold. First, we are primarily interested in $L_2[a, b]$, which is an inner product space. Secondly, the proof is easier if you can use orthogonality.)

5. (10 points) Let $\{\phi_n\}$ be an orthogonal sequence in a Hilbert space, $\mathcal{H}$. Show that 0 is the only vector orthogonal to all ϕ_n if and only if $\{\phi_n\}$ is a basis for $\mathcal{H}$. Hint: for any $x \in \mathcal{H}$, consider the function $y_x = x - \sum \frac{\langle x, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n$.

6. (20 points) Let $\{p_i\}$ be an orthonormal basis for $L_2[a, b]$. Consider $L_2([a, b]^2)$ with inner product

$$\langle f, g \rangle = \int_a^b \int_a^b f(x, y) g(x, y) \, dx \, dy,$$

for $f, g \in L_2([a, b]^2)$.

   (a) Show that $\{p_i(x) p_j(y)\}$ is an orthonormal sequence in $L_2([a, b]^2)$.
   (b) Show that $\{p_i(x) p_j(y)\}$ is a basis for $L_2([a, b]^2)$. Hint: see problem 5.