Tufts University
Department of Mathematics
Math 250-03 Homework 7

Due: Thursday, November 15, at 3:00 p.m. (in class).

1. (10 points) Let $\mathcal{H}$ be a Hilbert Space, and $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to \mathcal{H}$ be operators with adjoints A^* and B^*. Prove that
   (a) $(A + B)^* = A^* + B^*$
   (b) $(AB)^* = B^* A^*$

2. (20 points) A slight refinement on the definition of an adjoint given in Griffel is as follows. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces with inner-products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Let $\mathcal{U} \subset \mathcal{H}_1$ and $\mathcal{V} \subset \mathcal{H}_2$. The adjoint of $A : \mathcal{U} \to \mathcal{H}_2$ is $A^* : \mathcal{V} \to \mathcal{H}_1$ if $\langle Au, v \rangle_2 = \langle u, A^* v \rangle_1$ for all $u \in \mathcal{U}$, $v \in \mathcal{V}$.

   Let $\mathcal{H}_1 = L_2([0,1]^3)$ and $\mathcal{H}_2 = \left(L_2([0,1]^3)\right)^3$, meaning that each element in $\mathcal{H}_2$ is a vector-field from $[0,1]^3$ into $\mathbb{R}^3$. The inner product on $\mathcal{H}_1$ is the usual one:

   $$\langle u, v \rangle_1 = \int_0^1 \int_0^1 \int_0^1 u(x,y,z) v(x,y,z) dz\, dy\, dx.$$

   The inner product on $\mathcal{H}_2$ is constructed from this:

   $$\langle \mathbf{u}, \mathbf{v} \rangle_2 = \langle u_1, v_1 \rangle_1 + \langle u_2, v_2 \rangle_1 + \langle u_3, v_3 \rangle_1,$$

   for $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, both in $\mathcal{H}_2$.

   A common subset of $\mathcal{H}_1$ is

   $$H_{1,0}([0,1]^3) = \left\{ u \in \mathcal{H}_1 \mid \nabla u \in \mathcal{H}_2, u = 0 \text{ on boundary of } [0,1]^3 \right\}.$$

   Two common subsets of $\mathcal{H}_2$ are

   $$H_0(div) = \left\{ \mathbf{u} \in \mathcal{H}_2 \mid \nabla \cdot \mathbf{u} \in \mathcal{H}_1, \mathbf{u} = \mathbf{0} \text{ on boundary of } [0,1]^3 \right\}$$
   $$H_0(curl) = \left\{ \mathbf{u} \in \mathcal{H}_2 \mid \nabla \times \mathbf{u} \in \mathcal{H}_2, \mathbf{u} = \mathbf{0} \text{ on boundary of } [0,1]^3 \right\}.$$

   (a) What is the adjoint of the gradient operator $\nabla : H_{1,0}([0,1]^3) \to \mathcal{H}_2$? What is the domain of ∇^*?
   (b) Show that the curl operator, $\nabla \times : H_0(curl) \to \mathcal{H}_2$ is self-adjoint in $\mathcal{H}_2$.

   Hint: Both of these are natural consequences of the divergence theorem and some vector calculus identities.

3. (10 points) Let A be a bounded linear operator on a Hilbert space, $\mathcal{H}$. Define $\mathcal{R}(A) = \{y \mid y = Ax \text{ for } x \in \mathcal{H}\}$ and $\mathcal{N}(A) = \{x \mid Ax = 0\}$. Show that $(\mathcal{R}(A))^\perp = \mathcal{N}(A^*)$.

4. (20 points) Show that the following are Sturm Liouville boundary conditions for a regular Sturm-Liouville equation.
   (a) $u(a) = u(b) = 0$

(b) $u'(a) = u'(b) = 0$

(c) $u(a) = u(b)$, $u'(a) = u'(b)$, when $p(a) = p(b)$

(d) $u(a) - u'(a) = 0$, $u(b) + 2u'(b) = 0$.

5. (20 points)

(a) Rewrite Poisson's Equation, $-\partial_{xx}u - \partial_{yy}u = f$ on the unit disk in polar coordinates, $r^2 = x^2 + y^2$, $\theta = \arctan(y/x)$.

(b) Separate variables on the equation you get in part (a) with $f = 0$, writing $u(r, \theta) = R(r)\Theta(\theta)$. What equations do $R(r)$ and $\Theta(\theta)$ satisfy? Knowing that $\Theta(\theta)$ should be periodic with period 2π ($\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$), what does this say about the eigenvalues?

(c) Find the Sturm-Liouville eigenvalue problem satisfied by $R(r)$. Is the equation singular or regular? Do Dirichlet boundary conditions on the boundary of the unit disk lead to Sturm-Liouville boundary conditions?

6. (20 points)

(a) Separate variables on

$$\frac{1}{\sqrt{1-x^2}} u_t = \frac{\partial}{\partial x}\left(\sqrt{1-x^2}\frac{\partial}{\partial x}u\right), \text{ for } -1 < x < 1, t > 0$$

writing $u(x, t) = X(x)T(t)$. What equations do $X(x)$ and $T(t)$ satisfy? Assume the solution decays in time. What does this say about the eigenvalues?

(b) Find the Sturm-Liouville eigenvalue problem satisfied by $X(x)$. Is the equation singular or regular?

(c) Writing $x = \cos(y)$, rewrite the equation for $X(x)$ in terms of y. Solve this equation for $\hat{X}(y)$, then find $X(x) = \hat{X}(\arccos(x))$.