Tufts University
Department of Mathematics
Math 250-03 Homework 4

Due: Thursday, October 11, at 3:00 p.m. (in class).

1. (20 points) Recall the $\varepsilon - \delta$ definition of a continuous function: The function $f : [a, b] \to \mathbb{R}$ is continuous at x_0 if, for any ε, there exists a δ, such that if $x \in [a, b]$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. The function f is continuous on $[a, b]$ if it is continuous at each point in $[a, b]$.

   (a) Show that if f and g are continuous on $[a, b]$, then $(f+g)(x) = f(x)+g(x)$ is continuous on $[a, b]$.

   (b) Show that if f is continuous on $[a, b]$, then $(kf)(x) = kf(x)$ is continuous on $[a, b]$ for any constant $k \in \mathbb{R}$.

2. (20 points) Show that the function

$$f(x) = \begin{cases} 0 & x \leq 1/2 \\ 1 & x > 1/2 \end{cases}$$

   is in $L_2[0, 1]$. This shows that there are discontinuous functions in $L_2[0, 1]$.

3. (20 points) Define $C_1[a, b]$ to be the space of continuous functions on $[a, b]$ with continuous derivatives on $[a, b]$, and define the norm

$$\|f\|_1 = \left(\int_a^b (f(x))^2 dx + \int_a^b (f'(x))^2 dx \right)^{1/2}.$$

   Prove that $\| \cdot \|_1$ is a norm on $C_1[a, b]$. Note that you will need to prove the Cauchy-Schwarz inequality for this norm.

   Just as $L_2[a, b]$ is defined by taking the limits of sequences in $C[a, b]$, we can define the space $H_1[a, b]$ by taking limits of sequences in $C_1[a, b]$; this space is called a Sobolev space, and plays an important role in the analysis of elliptic PDEs.

4. (10 points) Let $\mathcal{V}$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and induced norm $\|v\|^2 = \langle v, v \rangle$.

   (a) Prove that

$$\|u + v\|^2 + \|u - v\|^2 = 2 \left(\|u\|^2 + \|v\|^2 \right)$$

   for any $u, v \in \mathcal{V}$.

   (b) The above equality is called the "parallelogram law". Explain why.

   (c) Prove that

$$4\langle u, v \rangle = \|u + v\|^2 - \|u - v\|^2$$

   for all $u, v \in \mathcal{V}$.

5. (30 points) Let

$$\mathcal{V} = \left\{ f : [0, \infty) \to \mathbb{R} \mid \int_0^\infty |f(t)|^2 e^{-t} dt < \infty \right\}.$$

(a) Show that $\langle f, g \rangle = \int_0^\infty f(t)g(t)e^{-t}dt$ defines an inner product on $\mathcal{V}$, again interpreted in terms of equivalence classes of functions that differ on sets of measure zero.

(b) Show that $\left\{L_n(t) = (e^t/n!)\frac{d^n}{dt^n}(t^n e^{-t})\right\}$ is an orthogonal set in $\mathcal{V}$ with this inner product. Hint: first show that $L_n(t)$ is a polynomial of degree n.