MATH 2260 (Ordinary Differential Equations I) — Winter 2015
Homework #6

Due Date: Wednesday, March 11th, in class or in marking box #31 by 5:00 PM. You must show all work to receive credit.

1. (10 points each) Find particular solutions for each of the following equations.

   (a) $y'' - 2y' + y = 14x^{3/2}e^x$

   (b) $y'' - 3y' + 2y = 4/(1 + e^{-x})$

   (c) $x^2y'' - 2xy' + (x^2 + 2)y = x^3\cos(x)$. Note that $y(x) = c_1x\cos(x) + c_2x\sin(x)$ is the general solution of the related homogeneous problem. (You don't need to verify that.)

2. (10 points) Consider the unforced, undamped spring system, modelled by $m\frac{d^2x}{dt^2} + kx = 0$, and take $\omega = \sqrt{k/m}$. Given initial conditions, $x(0) = x_0$ and $x'(0) = v_0$ not both zero, show that $x(t) = A\cos(\omega t - \alpha)$ for $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$ and α defined by $\cos\alpha = \frac{x_0}{A}$, $\sin\alpha = \frac{v_0}{\omega A}$.

3. (10 points) Consider the unforced, underdamped spring system, modelled by $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$, with $b^2 - 4mk < 0$ and take $\sigma = \frac{b}{2m}$ and $\omega = \frac{\sqrt{4mk-b^2}}{2m}$. Given initial conditions, $x(0) = x_0$ and $x'(0) = v_0$ not both zero, show that $x(t) = Ae^{-\sigma t}\cos(\omega t - \alpha)$ for $A = \sqrt{x_0^2 + \left(\frac{v_0+\sigma x_0}{\omega}\right)^2}$ and α defined by $\cos\alpha = \frac{x_0}{A}$, $\sin\alpha = \frac{v_0+\sigma x_0}{\omega A}$.

4. (20 points) Consider the unforced, overdamped spring system, modelled by $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$, with $b^2 - 4mk > 0$ and take $\sigma_1 = \frac{b-\sqrt{b^2-4mk}}{2m}$ and $\sigma_2 = \frac{b+\sqrt{b^2-4mk}}{2m}$.

   (a) Given initial conditions, $x(0) = x_0$ and $x'(0) = v_0$ not both zero, show that

   $$x(t) = \left(\frac{\sigma_2x_0 + v_0}{\sigma_2 - \sigma_1}\right)e^{-\sigma_1t} - \left(\frac{\sigma_1x_0 + v_0}{\sigma_2 - \sigma_1}\right)e^{-\sigma_2t}.$$

   (b) Show that there is at most one time, t_1, where this solution satisfies $x(t_1) = 0$ and find an expression for t_1.

   (c) Show that there is at most one time, $t_2 > 0$, where $x(t)$ has a local extremum.

5. (5 points each) Compute the following determinants:

   (a) $\det\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 0 & 5 & 0 \\ 0 & 1 & 3 & 7 \\ 0 & 5 & 0 & 5 \end{bmatrix}$ (b) $\det\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & -3 & 3 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 5 & 3 & 5 & 4 \\ 1 & -2 & 2 & 6 & 5 \end{bmatrix}$

6. (10 points) We say that a matrix is *upper triangular* if all entries below the *diagonal* (entries a_{ii} for $i = 1, \dots, n$) are zero, and that it is *lower triangular* if all entries above the diagonal are zero. Use expansion by minors to show that the determinant of a matrix that is either upper or lower triangular is the product of the diagonal entries. In other words, show that

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \det \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

7. (10 points) Check that $y_1(x) = x + 1$, $y_2(x) = 1 + x^2$, and $y_3(x) = x^2 - x$ are solutions of the equation $y''' = 0$ for all x. Is $\{y_1, y_2, y_3\}$ a fundamental set?