MATH 2260 (Ordinary Differential Equations I) — Fall 2014
Homework #8

Due Date: Thursday, November 27, in class or in marking box #59 by 5:00 PM. You must show all work to receive credit.

1. (10 points) Consider the unforced, undamped spring system, modelled by $m\frac{d^2x}{dt^2} + kx = 0$, and take $\omega = \sqrt{k/m}$. Given initial conditions, $x(0) = x_0$ and $x'(0) = v_0$ not both zero, show that $x(t) = A\cos(\omega t - \alpha)$ for $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$ and α defined by $\cos\alpha = \frac{x_0}{A}$, $\sin\alpha = \frac{v_0}{\omega A}$.

2. (10 points) Consider the unforced, underdamped spring system, modelled by $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$, with $b^2 - 4mk < 0$ and take $\sigma = \frac{b}{2m}$ and $\omega = \frac{\sqrt{4mk-b^2}}{2m}$. Given initial conditions, $x(0) = x_0$ and $x'(0) = v_0$ not both zero, show that $x(t) = Ae^{-\sigma t}\cos(\omega t - \alpha)$ for $A = \sqrt{x_0^2 + \left(\frac{v_0+\sigma x_0}{\omega}\right)^2}$ and α defined by $\cos\alpha = \frac{x_0}{A}$, $\sin\alpha = \frac{v_0+\sigma x_0}{\omega A}$.

3. (20 points) Consider the unforced, overdamped spring system, modelled by $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$, with $b^2 - 4mk > 0$ and take $\sigma_1 = \frac{b-\sqrt{b^2-4mk}}{2m}$ and $\sigma_2 = \frac{b+\sqrt{b^2-4mk}}{2m}$.

   (a) Given initial conditions, $x(0) = x_0$ and $x'(0) = v_0$ not both zero, show that
   $$x(t) = \left(\frac{\sigma_2 x_0 + v_0}{\sigma_2 - \sigma_1}\right)e^{-\sigma_1 t} - \left(\frac{\sigma_1 x_0 + v_0}{\sigma_2 - \sigma_1}\right)e^{-\sigma_2 t}.$$

   (b) Show that there is at most one time, t_1, where this solution satisfies $x(t_1) = 0$ and find an expression for t_1.

   (c) Show that there is at most one time, $t_2 > 0$, where $x(t)$ has a local extremum.

4. (5 points each) Note that these questions consider a mass, m, hanging under gravity (take $g = 9.8$ m/s^2) that stretches a spring to a new equilibrium position by length Δl. By Hooke's law, we can solve for the spring constant, k, as $mg = k\Delta l$.
   (a) Section 6.1, #3 (b) Section 6.1, #9 (c) Section 6.2, #15

5. (10 points each) Using the definition, compute the Laplace transforms of the following functions.
   (a) $f(t) = te^{3t}$ (b) $f(t) = e^t\sin(2t)$

6. (5 points each) Compute the Laplace transforms of the following functions. (You need not use the definition, but must show all work to receive credit.)

   (a) $f(t) = \cosh(\lambda t)$

   (b) $f(t) = t^2 - 7 + \cos 2t$

   (c) $f(t) = e^{2t+3}$

   (d) $f(t) = \sin\left(t + \frac{\pi}{6}\right)$

   (e) $f(t) = (t+1)(t+2)$