MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Worksheet	**MATH 3202**	Spring 2019

SOLUTIONS

1. (a) We have

$$P(x,y,z) = z^2 - y^2, \quad Q(x,y,z) = z^2 - x^2, \quad R(x,y,z) = y^2 - x^2.$$

Hence

$$\begin{aligned}\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} &= 2y - 2z \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} &= 2z - (-2x) = 2x + 2z \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= (-2x) - (-2y) = 2y - 2x.\end{aligned}$$

Thus

$$\operatorname{curl}(\mathbf{F}) = \langle 2y - 2z,\ 2x + 2z,\ 2y - 2x \rangle.$$

Since $\operatorname{curl}(\mathbf{F}) \neq \mathbf{0}$, $\mathbf{F}$ is <u>not</u> conservative. Furthermore,

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial R}{\partial z} = 0$$

so

$$\operatorname{div}(\mathbf{F}) = 0 + 0 + 0 = 0.$$

(b) We have

$$P(x,y,z) = z\sin(y), \quad Q(x,y,z) = xz\cos(y), \quad R(x,y,z) = x\sin(y).$$

Hence

$$\begin{aligned}\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} &= x\cos(y) - x\cos(y) = 0 \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} &= \sin(y) - \sin(y) = 0 \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= z\cos(y) - z\cos(y) = 0.\end{aligned}$$

Thus

$$\operatorname{curl}(\mathbf{F}) = \langle 0,\ 0,\ 0 \rangle = \mathbf{0}$$

and so $\mathbf{F}$ is conservative. Furthermore,

$$\frac{\partial P}{\partial x} = \frac{\partial R}{\partial z} = 0 \quad \text{and} \quad \frac{\partial Q}{\partial y} = -xz\sin(y)$$

so

$$\operatorname{div}(\mathbf{F}) = 0 - xz\sin(y) + 0 = -xz\sin(y).$$

(c) We have

$$P(x,y,z) = \frac{y}{xz}, \quad Q(x,y,z) = \frac{y\ln(x)}{z}, \quad R(x,y,z) = -\frac{y^2\ln(x)}{2z^2}.$$

Hence

$$\begin{aligned}
\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} &= -\frac{y\ln(x)}{z^2} - \left(-\frac{y\ln(x)}{z^2}\right) = 0\\
\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} &= -\frac{y}{xz^2} - \left(-\frac{y^2}{2xz^2}\right) = \frac{y^2-2y}{2xz^2}\\
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= \frac{y}{xz} - \frac{1}{xz} = \frac{y-1}{xz}.
\end{aligned}$$

Thus

$$\text{curl}(\mathbf{F}) = \left\langle 0,\ \frac{y^2-2y}{2xz^2},\ \frac{y-1}{xz} \right\rangle.$$

Since $\text{curl}(\mathbf{F}) \neq \mathbf{0}$, $\mathbf{F}$ is <u>not</u> conservative. Furthermore,

$$\frac{\partial P}{\partial x} = -\frac{y}{x^2z}, \quad \frac{\partial Q}{\partial y} = \frac{\ln(x)}{z}, \quad \frac{\partial R}{\partial z} = \frac{y^2\ln(x)}{z^3}$$

so

$$\text{div}(\mathbf{F}) = -\frac{y}{x^2z} + \frac{\ln(x)}{z} + \frac{y^2\ln(x)}{z^3} = \frac{x^2z^2\ln(x) + x^2y^2\ln(x) - yz^2}{x^2z^3}.$$

2. Omitting the dependences on x, y and z for clarity, we assume that $\mathbf{F} = \langle P,\ Q,\ R\rangle$ and so

$$g\mathbf{F} = \langle gP,\ gQ,\ gR\rangle.$$

Thus

$$\begin{aligned}
\text{div}(g\mathbf{F}) - g\,\text{div}(\mathbf{F}) &= \frac{\partial}{\partial x}(gP) + \frac{\partial}{\partial y}(gQ) + \frac{\partial}{\partial z}(gR) - g\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\\
&= g\frac{\partial P}{\partial x} + P\frac{\partial g}{\partial x} + g\frac{\partial Q}{\partial y} + Q\frac{\partial g}{\partial y} + g\frac{\partial R}{\partial z} + R\frac{\partial g}{\partial z} - g\frac{\partial P}{\partial x} - g\frac{\partial Q}{\partial y} - g\frac{\partial R}{\partial z}\\
&= P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} + R\frac{\partial g}{\partial z}.
\end{aligned}$$

Futhermore, we observe that

$$\begin{aligned}
\mathbf{F}\cdot\text{div}(g) &= \langle P,\ Q,\ R\rangle \cdot \left\langle \frac{\partial g}{\partial x},\ \frac{\partial g}{\partial y},\ \frac{\partial g}{\partial z}\right\rangle\\
&= P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} + R\frac{\partial g}{\partial z}
\end{aligned}$$

as well. Hence

$$\text{div}(g\mathbf{F}) - g\,\text{div}(\mathbf{F}) = \mathbf{F}\cdot\nabla g,$$

as required.

3. To evaluate the line integral directly, we first observe that S is shaped like a triangle with vertices $(4,0,0)$, $(0,1,0)$ amd $(0,0,8)$. Thus its boundary consists of the three lines joining each pair of vertices.

First consider C_1 from $(4,0,0)$ to $(0,1,0)$. The line is described by the function

$$\mathbf{r}(t) = \langle 4 - 4t,\ t,\ 0\rangle$$

for $0 \le t \le 1$. Thus

$$\mathbf{F}(\mathbf{r}(t)) = \langle 4t - 4t^2,\ -t,\ 8t - 8\rangle$$

and

$$\mathbf{r}'(t) = \langle -4,\ 1,\ 0\rangle$$

so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16t + 16t^2 - t = 16t^2 - 17t.$$

Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (16t^2 - 17t)\, dt = -\frac{19}{6}.$$

Next consider C_2 from $(0,1,0)$ to $(0,0,8)$. The line is described by the function

$$\mathbf{r}(t) = \langle 0,\ 1 - t,\ 8t\rangle$$

for $0 \le t \le 1$. Thus

$$\mathbf{F}(\mathbf{r}(t)) = \langle 0,\ 9t - 1,\ 0\rangle$$

and

$$\mathbf{r}'(t) = \langle 0,\ -1,\ 8\rangle$$

so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 1 - 9t.$$

Thus

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (1 - 9t)\, dt = -\frac{7}{2}.$$

Lastly consider C_3 from $(0,0,8)$ to $(4,0,0)$. The line is described by the function

$$\mathbf{r}(t) = \langle 4t,\ 0,\ 8 - 8t\rangle$$

for $0 \le t \le 1$. Thus

$$\mathbf{F}(\mathbf{r}(t)) = \langle 0,\ 8 - 8t,\ -8t\rangle$$

and

$$\mathbf{r}'(t) = \langle 4,\ 0,\ -8\rangle$$

so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 64t.$$

Thus

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (64t)\, dt = 32.$$

Hence

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \frac{76}{3}.$$

Alternatively, we have

$$\text{curl}(\mathbf{F}) = \langle -1,\ 2,\ -x\rangle$$

and we observe that the plane can be written as the function

$$z = 8 - 2x - 8y.$$

Thus

$$\iint\limits_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint\limits_D [2(-1) + 8(2) + (-x)]\, dA = \iint\limits_D (14 - x)\, dA$$

where D is the region of integration. The projection of $2x + 8y + z = 8$ onto the xy-plane is the line $2x + 8y = 8$ or $y = 1 - \frac{1}{4}x$. Since we are only interested in the first octant, then, D is bounded by $0 \le y \le 1 - \frac{1}{4}x$ and $0 \le x \le 4$. We can therefore write

$$\begin{aligned}\iint\limits_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} &= \int_0^4 \int_0^{1-\frac{1}{4}x} (14 - x)\, dy\, dx \\ &= \int_0^4 \Big[y(14 - x)\Big]_{y=0}^{y=1-\frac{1}{4}x} dx \\ &= \int_0^4 \left(\frac{1}{4}x^2 - \frac{9}{2}x + 14\right) dx \\ &= \frac{76}{3}.\end{aligned}$$

4. In order to evaluate the circulation directly, we would have to compute four individual line integrals to represent each side of the rectangle. Instead, we can observe that the surface S bounded by C is part of the plane $z = 4$ for $0 \le x \le 3$ and $0 \le y \le 2$. Furthermore,

$$\text{curl}(\mathbf{F}) = \langle 0,\ -3,\ 2x\rangle$$

so, since an upward-pointing normal to S is given by the vector $\langle 0,\ 0,\ 1\rangle$ we have

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} &= \int_0^3 \int_0^2 2x\, dy\, dx \\ &= \int_0^3 \Big[x^2\Big]_{y=0}^{y=2} dx \\ &= 4\int_0^3 dx \\ &= 4\Big[x\Big]_0^3 \\ &= 12.\end{aligned}$$

5. The hemisphere is bounded by the curve ∂S comprising the circle $(x-2)^2 + y^2 = 4$. Thus it can be parametrised by the function

$$\mathbf{r}(t) = \langle 2\cos(t) + 2,\ 2\sin(t),\ 0\rangle$$

for $0 \leq t \leq 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle 0,\ 2\sin(t),\ 2\cos(t) + 2\rangle$$

and

$$\mathbf{r}'(t) = \langle -2\sin(t),\ 2\cos(t),\ 0\rangle$$

so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 4\sin(t)\cos(t).$$

Hence

$$\begin{aligned}\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} 4\sin(t)\cos(t)\, dt \\ &= \Big[2\sin^2(t)\Big]_0^{2\pi} \\ &= 0.\end{aligned}$$

6. The boundary of E consists of the paraboloid, which we will call S_1, together with the disc $x^2 + y^2 = 1$, which we will call S_2. Given the circular nature of S_2, we will work in polar coordinates.

Thus S_1 can be parametrised by the function

$$\mathbf{R}(r, \theta) = \langle r\cos(\theta),\ r\sin(\theta),\ r^2\cos^2(\theta) + r^2\sin^2(\theta)\rangle = \langle r\cos(\theta),\ r\sin(\theta),\ r^2\rangle$$

where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Then

$$\mathbf{R}_r = \langle \cos(\theta),\ \sin(\theta),\ 2r\rangle$$

and

$$\mathbf{R}_\theta = \langle -r\sin(\theta),\ r\cos(\theta),\ 0\rangle$$

so

$$\mathbf{R}_\theta \times \mathbf{R}_r = \langle 2r^2\cos(\theta),\ 2r^2\sin(\theta),\ -r\rangle.$$

(Note that we need this normal rather than $\mathbf{R}_r \times \mathbf{R}_\theta$ to ensure that it points outward which, given the bowl-like shape of the paraboloid, would require a negative z-component.) Next,

$$\mathbf{F} = \langle r\cos(\theta),\ r\sin(\theta),\ r^4\cos(\theta)\sin(\theta)\rangle$$

so

$$\mathbf{F} \cdot (\mathbf{R}_r \times \mathbf{R}_\theta) = 2r^3\cos^2(\theta) + 2r^3\sin^2(\theta) - r^5\cos(\theta)\sin(\theta) = 2r^3 - r^5\cos(\theta)\sin(\theta).$$

Now we have

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 [2r^3 - r^5\cos(\theta)\sin(\theta)]\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{2}r^4 - \frac{1}{6}r^6\cos(\theta)\sin(\theta)\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{2} - \frac{1}{6}\cos(\theta)\sin(\theta)\right] d\theta$$
$$= \left[\frac{1}{2}\theta - \frac{1}{12}\sin^2(\theta)\right]_0^{2\pi}$$
$$= \pi.$$

Next, S_2 can be parametrised by the function

$$\mathbf{R}(r,\theta) = \langle r\cos(\theta),\ r\sin(\theta),\ 1\rangle$$

where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Then

$$\mathbf{R}_r = \langle \cos(\theta),\ \sin(\theta),\ 0\rangle$$

and

$$\mathbf{R}_\theta = \langle -r\sin(\theta),\ r\cos(\theta),\ 0\rangle$$

so

$$\mathbf{R}_r \times \mathbf{R}_\theta = \langle 0,\ 0,\ r\rangle.$$

Furthermore,

$$\mathbf{F} = \langle r\cos(\theta),\ r\sin(\theta),\ r^2\cos(\theta)\sin(\theta)\rangle$$

so

$$\mathbf{F}\cdot(\mathbf{R}_r \times \mathbf{R}_\theta) = r^2\cos(\theta)\sin(\theta).$$

Now we have

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 [r^2\cos(\theta)\sin(\theta)]\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{3}r^3\cos(\theta)\sin(\theta)\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{3}\cos(\theta)\sin(\theta)\right] d\theta$$
$$= \left[\frac{1}{6}\sin^2(\theta)\right]_0^{2\pi}$$
$$= 0.$$

Hence

$$\iint \partial E \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \pi + 0 = \pi.$$

Alternatively, we have

$$\operatorname{div}(\mathbf{F}) = 1 + 1 + xy = xy + 2.$$

In cylindrical coordinates, this becomes

$$\operatorname{div}(\mathbf{F}) = r^2\cos(\theta)\sin(\theta) + 2.$$

The region of integration is defined by $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, and because the paraboloid becomes the curve $z = r^2$, we also have $r^2 \le z \le 1$. Hence

$$\begin{aligned}
\iiint_E \operatorname{div}(\mathbf{F})\,dV &= \int_0^{2\pi}\int_0^1\int_{r^2}^1 [r^2\cos(\theta)\sin(\theta) + 2]\cdot r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1\int_{r^2}^1 [r^3\cos(\theta)\sin(\theta) + 2r]\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \Big[r^3 z\cos(\theta)\sin(\theta) + 2rz\Big]_{z=r^2}^{z=1}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 [r^3\cos(\theta)\sin(\theta) + 2r - r^5\cos(t)\sin(t) - 2r^3]\,dr\,d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{4}r^4\cos(\theta)\sin(\theta) + r^2 - \frac{1}{6}r^6\cos(\theta)\sin(\theta) - \frac{1}{2}r^4\right]_{r=0}^{r=1} d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{12}\cos(\theta)\sin(\theta) + \frac{1}{2}\right] d\theta \\
&= \left[\frac{1}{24}\sin^2(\theta) + \frac{1}{2}\theta\right]_0^{2\pi} \\
&= \pi.
\end{aligned}$$

7. In order to evaluate the surface integral directly, we would have to recognise that S is the union of six surfaces, each of which would have to be separately parametrised and the corresponding surface integral evaluated. Instead, we simply compute

$$\operatorname{div}(\mathbf{F}) = 3y^2 z$$

and observe that S bounds the cube for which $0 \le x \le 2$, $0 \le y \le 3$ and $0 \le z \le 4$. Hence, by the Divergence Theorem,

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 3y^2z\,dV \\
&= \int_0^2\int_0^3\int_0^4 3y^2z\,dz\,dy\,dx \\
&= \int_0^2\int_0^3 \left[\frac{3}{2}y^2z^2\right]_{z=0}^{z=4} dy\,dx \\
&= \int_0^2\int_0^3 24y^2\,dy\,dx \\
&= \int_0^2 \left[8y^3\right]_{y=0}^{y=3} dx \\
&= \int_0^2 216\,dx \\
&= \left[216x\right]_0^2 \\
&= 432.
\end{aligned}$$

8. By the Divergence Theorem, we can write

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}(\mathbf{F})\,dV.$$

But

$$\operatorname{div}(\mathbf{F}) = \operatorname{div}(\operatorname{curl}(\mathbf{G})) = 0.$$

Hence

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0.$$

9. The constraint is described by the function $g(x,y) = 3x^2 + 4y^2$ so

$$\nabla f = \langle 1,\ 2\rangle \quad \text{and} \quad \nabla g = \langle 6x,\ 8y\rangle.$$

Hence we require

$$\langle 1,\ 2\rangle = \lambda\langle 6x,\ 8y\rangle$$

and so $1 = 6\lambda x$ and $2 = 8\lambda y$. From the first equation,

$$\lambda = \frac{1}{6x}$$

and therefore, substituting into the second equation, we have

$$2 = 8\left(\frac{1}{6x}\right)y \quad \Longrightarrow \quad y = \frac{3}{2}x.$$

Thus the constraint can be written

$$\begin{aligned} 3x^2 + 4\left(\frac{3}{2}x\right)^2 &= 3 \\ 12x^2 &= 3 \\ x^2 &= \frac{1}{4} \\ x &= \pm\frac{1}{2}. \end{aligned}$$

When $x = \frac{1}{2}$, $y = \frac{3}{4}$ and so $f\left(\frac{1}{2}, \frac{3}{4}\right) = 2$. When $x = -\frac{1}{2}$, $y = -\frac{3}{4}$ and thus $f\left(-\frac{1}{2}, -\frac{3}{4}\right) = -2$. Hence we conclude that the minimum value of the function is -2.

10. Since we want to minimise the distance from the desired point $P(x, y, z)$ to the origin, we wish to minimise the function

$$f(x, y, z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

However, we can make our work simpler by recognising that this is tantamount to minimising the function

$$F(x, y, z) = [f(x, y, z)]^2 = x^2 + y^2 + z^2.$$

Either way, the constraint is described by the function

$$g(x, y, z) = x + 2y + 3z.$$

Thus

$$\nabla F = \langle 2x,\ 2y,\ 2z\rangle \quad \text{and} \quad \nabla g = \langle 1,\ 2,\ 3\rangle.$$

Now we have

$$\langle 2x,\ 2y,\ 2z\rangle = \lambda\langle 1,\ 2,\ 3\rangle.$$

Thus $2x = \lambda$, $2y = 2\lambda$ so $y = \lambda$, and $2z = 3\lambda$. This means that $x = \frac{1}{2}y$ and $z = \frac{3}{2}y$. Substituting back into the constraint, we find

$$\begin{aligned} \frac{1}{2}y + 2y + \frac{9}{2}y &= 7 \\ 7y &= -14 \\ y &= -2. \end{aligned}$$

Thus $x = -1$ and $z = -3$ and so the point P has coordinates $(-1, -2, -3)$.

11. Let ℓ, w and h be the length, width and height of the box. We wish to maximise the function $f(\ell, w, h) = 2\ell h + 2wh + \ell w$ where the constraint is described by $g(\ell, w, h) = 2\ell + 2w + 2h$. Thus

$$\nabla f = \langle 2h + w,\ 2h + \ell,\ 2\ell + 2w\rangle \quad \text{and} \quad \nabla g = \langle 2,\ 2,\ 2\rangle.$$

We therefore require

$$\langle 2h + w,\ 2h + \ell,\ 2\ell + 2w\rangle = \lambda\langle 2,\ 2,\ 2\rangle$$

and so $2h + w = 2\lambda$, $2h + \ell = 2\lambda$ and $2\ell + 2w = 2\lambda$ so $\ell + w = \lambda$. The first equation tells us that $w = 2\lambda - 2h$ and the second equation likewise yields $\ell = 2\lambda - 2h$. Substituting these into the third equation, we obtain

$$\begin{aligned}(2\lambda - 2h) + (2\lambda - 2h) &= \lambda \\ 3\lambda &= 4h \\ h &= \frac{3}{4}\lambda\end{aligned}$$

and therefore $w = \ell = \frac{1}{2}\lambda$. The constraint can now be written

$$\begin{aligned}2\left(\frac{1}{2}\lambda\right) + 2\left(\frac{1}{2}\lambda\right) + 2\left(\frac{3}{4}\lambda\right) &= 35 \\ \frac{7}{2}\lambda &= 35 \\ \lambda &= 10.\end{aligned}$$

Thus $w = \ell = 5$ cm and $h = \frac{15}{2}$ cm, which means that the maximum surface area is $\frac{575}{2} = 287.5$ cm^2.