MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 8 **MATH 3202** SPRING 2019

SOLUTIONS

[3] 1. (a) Here,

$$M(x,y) = 4xy^3 - 5 \quad \text{and} \quad N(x,y) = 6x^2y^2$$

so

$$\frac{\partial M}{\partial y} = 12xy^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 12xy^2.$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\mathbf{F}$ is conservative.

Next we must have

$$f(x,y) = \int (4xy^3 - 5)\,dx = 2x^2y^3 - 5x + C(y)$$

where C is an arbitrary function of y. Then

$$f_y(x,y) = 6x^2y^2 + C'(y) = 6x^2y^2$$

and so

$$C'(y) = 0 \quad \Longrightarrow \quad C(y) = C.$$

Hence the potential function takes the form

$$f(x,y) = 2x^2y^3 - 5x + C.$$

[1] (b) Here,

$$M(x,y) = x\ln(y) + 1 \quad \text{and} \quad N(x,y) = y\ln(x) + y^2$$

so

$$\frac{\partial M}{\partial y} = \frac{x}{y} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{y}{x}.$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, $\mathbf{F}$ is <u>not</u> conservative.

[4] (c) Here,

$$M(x,y) = \sin(y) - \sin(x) \quad \text{and} \quad N(x,y) = x\cos(y) - \sin(y)$$

so

$$\frac{\partial M}{\partial y} = \cos(y) \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos(y).$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\mathbf{F}$ is conservative.

Next we must have

$$f(x,y) = \int [\sin(y) - \sin(x)]\,dx = x\sin(y) + \cos(x) + C(y).$$

Then

$$f_y(x,y) = x\cos(y) + C'(y) = x\cos(y) - \sin(y)$$

and so

$$C'(y) = -\sin(y) \quad \Longrightarrow \quad C(y) = \cos(y) + C.$$

Hence the potential function takes the form

$$f(x,y) = x\sin(y) + \cos(x) + \cos(y) + C.$$

[1] (d) We have

$$P(x,y,z) = xy^2z^3, \quad Q(x,y,z) = x^2yz^3, \quad R(x,y,z) = x^2y^2z^2.$$

Hence

$$P_y(x,y,z) = 2xyz^3 \quad \text{and} \quad Q_x(x,y,z) = 2xyz^3$$

and so $P_y = Q_x$. Next,

$$P_z(x,y,z) = 3xy^2z^2 \quad \text{and} \quad R_x(x,y,z) = 2xy^2z^2.$$

Since $P_z \neq R_x$, we can immediately conclude that $\mathbf{F}$ is <u>not</u> conservative. Note that we could observe that

$$Q_z(x,y,z) = 3x^2yz^2 \quad \text{and} \quad R_y(x,y,z) = 2x^2yz^2$$

so, again, since $Q_z \neq R_y$, we have another reason to deduce that $\mathbf{F}$ is not conservative.

[5] (e) We have

$$P(x,y,z) = xy^2z^2 + 2, \quad Q(x,y,z) = x^2yz^2 + 3, \quad R(x,y,z) = x^2y^2z + 4.$$

Thus

$$P_y(x,y,z) = 2xyz^2 \quad \text{and} \quad Q_x(x,y,z) = 2xyz^2$$

so $P_y = Q_x$. Next,

$$P_z(x,y,z) = 2xy^2z \quad \text{and} \quad R_x(x,y,z) = 2xy^2z$$

so $P_z = R_x$. Finally,

$$Q_z(x,y,z) = 2x^2yz \quad \text{and} \quad R_y(x,y,z) = 2x^2yz$$

so $Q_z = R_y$. Since all three conditions are met, it must be that $\mathbf{F}$ is conservative.
Now we have

$$f(x,y,z) = \int (xy^2z^2 + 2)\,dx = \frac{1}{2}x^2y^2z^2 + 2x + C(y,z)$$

where $C(y,z)$ is an arbitrary function of y and z. Then

$$f_y(x,y,z) = x^2yz^2 + C_y(y,z) = x^2yz^2 + 3 \quad \Longrightarrow \quad C_y(y,z) = 3.$$

This means that

$$C(y,z) = \int 3\,dy = 3y + C(z)$$

and so

$$f(x,y,z) = \frac{1}{2}x^2y^2z^2 + 2x + 3y + C(z).$$

Finally,

$$f_z(x,y,z) = x^2y^2z + C'(z) = x^2y^2z + 4 \quad \Longrightarrow \quad C'(z) = 4 \quad \Longrightarrow \quad C(z) = 4z + C.$$

Hence the potential function has the form

$$f(x,y,z) = \frac{1}{2}x^2y^2z^2 + 2x + 3y + 4z + C.$$

[1] 2. We immediately have that

$$\int_C \nabla f \cdot d\mathbf{r} = f(2,0) - f(0,2) = 0 - 0 = 0.$$

This agrees with our result (obtained from first principles) on Assignment 7, Question #5(b).

[7] 3. We are given that $\mathbf{F}$ is conservative (although this could easily be checked) so we must first find its potential function. We have

$$f(x,y,z) = \int (2x - 3z)\,dx = x^2 - 3xz + C(y,z)$$

so

$$f_y(x,y,z) = C_y(y,z) = 3y^2 \quad \Longrightarrow \quad C(y,z) = y^3 + C(z).$$

Now

$$f(x,y,z) = x^2 - 3xz + y^3 + C(z)$$

so

$$f_z(x,y,z) = -3x + C'(z) = -3x \quad \Longrightarrow \quad C'(z) = 0 \quad \Longrightarrow \quad C(z) = C.$$

Hence the potential function has the form

$$f(x,y,z) = x^2 - 3xz + y^3 + C$$

and since any function from this family will satisfy the Fundamental Theorem of Line Integrals, we will choose $C = 0$. Thus

$$f(x,y,z) = x^2 - 3xz + y^3.$$

(a) The endpoints of C are $(3,0,0)$ when $t = 0$ and $(0,3,2\pi)$ when $t = \frac{\pi}{2}$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,3,2\pi) - f(3,0,0) = 27 - 9 = 18.$$

(b) We immediately have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(8,4,4) - f(2,-1,-3) = 32 - 21 = 11.$$

(c) We simply have to observe that C starts at $(0,0,0)$ and ends at $(e, 2e, e^2)$ so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(e,2e,e^2) - f(0,0,0) = (5e^3 + e^2) - 0 = 5e^3 + e^2.$$

[6] 4. (a) The curve C encloses a region D bounded above by the line $y = 2x$ and below by the line $y = 0$ on the interval from $x = 0$ to $x = 4$. Furthermore,

$$M(x,y) = y\sqrt{x^2+9} \quad \text{and} \quad N(x,y) = e^y$$

so

$$\frac{\partial M}{\partial y} = \sqrt{x^2+9} \quad \text{and} \quad \frac{\partial N}{\partial x} = 0.$$

Thus, by Green's Theorem,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_D \left(0 - \sqrt{x^2+9}\right)\, dA \\ &= -\int_0^4 \int_0^{2x} \sqrt{x^2+9}\, dy\, dx \\ &= -\int_0^4 \left[y\sqrt{x^2+9}\right]_{y=0}^{y=2x} dx \\ &= -\int_0^4 2x\sqrt{x^2+9}\, dx.\end{aligned}$$

Let $u = x^2 + 9$ so $du = 2x\,dx$. When $x = 0$, $u = 9$ and when $x = 4$, $u = 25$. The integral becomes

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= -\int_9^{25} \sqrt{u}\, du \\ &= -\left[\frac{2}{3}u^{\frac{3}{2}}\right]_9^{25} \\ &= -\frac{196}{3}.\end{aligned}$$

[6] (b) Given the circular symmetry of the curve C, we will work in polar coordinates, in which the unit circle corresponds to the equation $r = 1$. Thus the region D bounded by C is defined by $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Furthermore,

$$M(x,y) = -x^2 y \quad \text{and} \quad N(x,y) = xy^2$$

so

$$\frac{\partial M}{\partial y} = -x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = y^2.$$

Thus, by Green's Theorem,

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint\limits_D [y^2 - (-x^2)]\, dA \\
&= \iint\limits_D (x^2 + y^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 r^2 \cdot r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_{r=0}^{r=1} d\theta \\
&= \frac{1}{4} \int_0^{2\pi} d\theta \\
&= \frac{1}{4} \Big[\theta\Big]_0^{2\pi} \\
&= \frac{\pi}{2}.
\end{aligned}$$

[6] (c) The region D is bounded above by $y = \sqrt{x}$ and below by $y = x^2$ on the interval from $x = 0$ to $x = 1$. Furthermore,

$$M(x, y) = x + y^2 \quad \text{and} \quad N(x, y) = x^2 + 3xy$$

so

$$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x + 3y.$$

Thus, by Green's Theorem,

$$\begin{aligned}
\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \iint_D [(2x+3y) - 2y] \, dA \\
&= \iint_D (2x+y) \, dA \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} (2x+y) \, dy \, dx \\
&= \int_0^1 \left[2xy + \frac{1}{2}y^2 \right]_{y=x^2}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left[2x^{\frac{3}{2}} + \frac{1}{2}x - 2x^3 - \frac{1}{2}x^4 \right] dx \\
&= \left[\frac{4}{5}x^{\frac{5}{2}} + \frac{1}{4}x^2 - \frac{1}{2}x^4 - \frac{1}{10}x^5 \right]_0^1 \\
&= \frac{4}{5} + \frac{1}{4} - \frac{1}{2} - \frac{1}{10} \\
&= \frac{9}{20}.
\end{aligned}$$

[2] 5. (a) Setting

$$x = \frac{3t}{1+t^3} \quad \text{and} \quad y = \frac{3t^2}{1+t^3}$$

the left-hand side of the equation of the folium becomes

$$\begin{aligned}
x^3 + y^3 &= \left(\frac{3t}{1+t^3} \right)^3 + \left(\frac{3t^2}{1+t^3} \right)^3 \\
&= \frac{27t^3}{(1+t^3)^3} + \frac{27t^6}{(1+t^3)^3} \\
&= \frac{27t^3(1+t^3)}{(1+t^3)^3} \\
&= \frac{27t^2}{(1+t^3)^2}.
\end{aligned}$$

The right-hand side becomes

$$\begin{aligned}
3xy &= 3 \left(\frac{3t}{1+t^3} \right) \left(\frac{3t^2}{1+t^3} \right) \\
&= \frac{27t^3}{(1+t^3)^2}.
\end{aligned}$$

Hence the equation is satisfied under the parametrisation.

[8] (b) We cannot use a double integral to find the area A because the top and bottom branches of the loop are not (easily) written as functions. Instead, we apply Green's Theorem to write

$$A = \frac{1}{2} \oint_C \mathbf{F} \cdot d\mathbf{r}$$

for $\mathbf{F} = \langle -y,\ x \rangle$ and C defined by $\mathbf{r}(t)$ for $0 \le t < \infty$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle -\frac{3t^2}{1+t^3},\ \frac{3t}{1+t^3} \right\rangle$$

and

$$\mathbf{r}'(t) = \left\langle \frac{3-6t^3}{(1+t^3)^2},\ \frac{6t-3t^4}{(1+t^3)^2} \right\rangle.$$

Now we can write

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \left\langle -\frac{3t^2}{1+t^3},\ \frac{3t}{1+t^3} \right\rangle \cdot \left\langle \frac{3-6t^3}{(1+t^3)^2},\ \frac{6t-3t^4}{(1+t^3)^2} \right\rangle \\
&= \frac{18t^5 - 9t^2}{(1+t^3)^3} + \frac{18t^2 - 9t^5}{(1+t^3)^3} \\
&= \frac{9t^2 + 9t^5}{(1+t^3)^3} \\
&= \frac{9t^2(1+t^3)}{(1+t^3)^3} \\
&= \frac{9t^2}{(1+t^3)^2}.
\end{aligned}$$

Finally,

$$A = \frac{1}{2} \int_0^\infty \frac{9t^2}{(1+t^3)^2}\, dt = \frac{9}{2} \lim_{T\to\infty} \int_0^T \frac{t^2}{(1+t^3)^2}\, dt.$$

Let $u = 1 + t^3$ so $\frac{1}{3}\, du = t^2\, dt$. When $t = 0$, $u = 1$. When $t = T$, $u = 1 + T^3$. The integral becomes

$$\begin{aligned}
A &= \frac{3}{2} \lim_{T\to\infty} \int_1^{1+T^3} \frac{1}{u^2}\, du \\
&= \frac{3}{2} \lim_{T\to\infty} \left[-\frac{1}{u} \right]_1^{1+T^3} \\
&= \frac{3}{2} \lim_{T\to\infty} \left[-\frac{1}{1+T^3} + 1 \right] \\
&= \frac{3}{2}[0 + 1] \\
&= \frac{3}{2}.
\end{aligned}$$