MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 7 **MATH 3202** Spring 2019

SOLUTIONS

[5] 1. In cylindrical coordinates, the two paraboloids become $z = r^2$ and $z = 2 - r^2$. They intersect when

$$r^2 = 2 - r^2 \quad \Longrightarrow \quad 2r^2 = 2 \quad \Longrightarrow \quad r = \pm 1.$$

Neglecting the negative option (since it will be handled by our choice of θ) we therefore have $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Finally, $r^2 \le z \le 2 - r^2$. Thus

$$\begin{aligned} V &= \iiint_E dV \\ &= \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 \Big[rz \Big]_{z=r^2}^{z=2-r^2} dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 (2r - 2r^3)\, dr\, d\theta \\ &= \int_0^{2\pi} \left[r^2 - \frac{1}{2} r^4 \right]_{r=0}^{r=1} d\theta \\ &= \frac{1}{2} \int_0^{2\pi} d\theta \\ &= \frac{1}{2} \Big[\theta \Big]_0^{2\pi} \\ &= \pi. \end{aligned}$$

[6] 2. In the xy-plane, the region of integration is bounded by $0 \le x \le 2$ and $0 \le y \le \sqrt{2x - x^2}$. The last of these bounds is equivalent to the upper branch of the graph with equation

$$y = \sqrt{2x - x^2} \quad \Longrightarrow \quad x^2 - 2x + y^2 = 0 \quad \Longrightarrow \quad (x-1)^2 + y^2 = 1,$$

namely a circle of radius 1 centred at the point $(1, 0)$. In cylindrical coordinates, it becomes

$$r \sin(\theta) = \sqrt{2r\cos(\theta) - r^2\cos^2(\theta)}$$

so squaring both sides and rearranging yields

$$r^2\cos^2(\theta) + r^2\sin^2(\theta) = 2r\cos(\theta) \quad \Longrightarrow \quad r^2 = 2r\cos(\theta) \quad \Longrightarrow \quad r = 2\cos(\theta).$$

Hence $0 \le r \le 2\cos(\theta)$. To traverse the upper semi-circle, we require only $0 \le \theta \le \frac{\pi}{2}$, since $r(0) = 2$ and $r(\frac{\pi}{2}) = 0$. Finally, we have $0 \le z \le \sqrt{x^2+y^2}$ which, in cylindrical coordinates, becomes $0 \le z \le r$. Since the integrand $\sqrt{x^2+y^2} = r$ we obtain

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^{\sqrt{x^2+y^2}} \sqrt{x^2+y^2}\, dz\, dy\, dx &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} \int_0^r r \cdot r\, dz\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} \int_0^r r^2\, dz\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} \Big[r^2 z\Big]_{z=0}^{z=r} dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^3\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4} r^4\right]_{r=0}^{r=2\cos(\theta)} d\theta \\
&= 4\int_0^{\frac{\pi}{2}} \cos^4(\theta)\, d\theta \\
&= 4\int_0^{\frac{\pi}{2}} \left[\frac{1+\cos(2\theta)}{2}\right]^2 d\theta \\
&= \int_0^{\frac{\pi}{2}} [1 + 2\cos(2\theta) + \cos^2(2\theta)]\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta)\right] d\theta \\
&= \left[\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta)\right]_0^{\frac{\pi}{2}} \\
&= \frac{3\pi}{4}.
\end{aligned}$$

[6] 3. In spherical coordinates, the hemisphere becomes $\rho = 1$ while the cone becomes

$$\rho^2 \sin^2(\phi)\cos^2(\theta) + \rho^2 \sin^2(\phi)\sin^2(\theta) = \rho^2\cos^2(\phi) \quad \Longrightarrow \quad \sin^2(\phi) = \cos^2(\phi).$$

Thus we must have $\tan^2(\phi) = 1$ and so $\tan(\phi) = \pm 1$. Since must have $0 \le \phi \le \pi$, the solutions are $\phi = \frac{\pi}{4}$ (the upper branch of the cone) and $\phi = \frac{3\pi}{4}$ (the lower branch of the cone). However, only the upper branch of the cone can intersect with the upper hemisphere. Hence $0 \le \rho \le 1$, $0 \le \phi \le \frac{\pi}{4}$ and, because both surfaces project onto complete circles in the

xy-plane, $0 \le \theta \le 2\pi$. Finally, the integrand $z^3 = \rho^3 \cos^3(\phi)$ so

$$\iiint_E z^3\, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^3 \cos^3(\phi) \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^5 \cos^3(\phi) \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \left[\frac{1}{6}\rho^6 \cos^3(\phi) \sin(\phi)\right]_{\rho=0}^{\rho=1} d\phi\, d\theta$$

$$= \frac{1}{6} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \cos^3(\phi) \sin(\phi)\, d\phi\, d\theta.$$

Let $u = \cos(\phi)$ so $-du = \sin(\phi)\, d\phi$. When $\phi = 0$, $u = 1$ and when $\phi = \frac{\pi}{4}$, $u = \frac{\sqrt{2}}{2}$. The integral becomes

$$\iiint_E z^3\, dV = -\frac{1}{6} \int_0^{2\pi} \int_1^{\frac{\sqrt{2}}{2}} u^3\, du\, d\theta$$

$$= -\frac{1}{6} \int_0^{2\pi} \left[\frac{1}{4} u^4\right]_{u=1}^{u=\frac{\sqrt{2}}{2}} d\theta$$

$$= \frac{1}{32} \int_0^{2\pi} d\theta$$

$$= \frac{1}{32} \Big[\theta\Big]_0^{2\pi}$$

$$= \frac{\pi}{16}.$$

[6] 4. In the xy-plane, the region is bounded by $-2 \le x \le 2$ and $0 \le y \le \sqrt{4 - x^2}$. The last of these bounds is equivalent to the upper branch of the graph with equation

$$y = \sqrt{4 - x^2} \quad \Longrightarrow \quad x^2 + y^2 = 4,$$

that is, a circle of radius 2 centred at the origin. Furthermore, we have $-\sqrt{4 - x^2 - y^2} \le z \le \sqrt{4 - x^2 - y^2}$, and these bounds correspond to the upper and lower branches of the graph of the equation

$$z^2 = 4 - x^2 - y^2 \quad \Longrightarrow \quad x^2 + y^2 + z^2 = 4,$$

a sphere of radius 2 centred at the origin. Hence $0 \le \rho \le 2$, $0 \le \phi \le \pi$, and because we are only interested in the upper semicircle in the xy-plane, $0 \le \theta \le \pi$. Lastly, the integrand becomes

$$\sqrt{x^2 + y^2} = \sqrt{\rho^2 \sin^2(\phi) \cos^2(\theta) + \rho^2 \sin^2(\phi) \cos^2(\theta)} = \sqrt{\rho^2 \sin^2(\phi)} = \rho \sin(\phi).$$

Thus

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_0^{\pi}\int_0^{\pi}\int_0^{2}\rho\sin(\phi)\cdot\rho^2\sin(\phi)\,dr\,d\phi\,d\theta$$
$$= \int_0^{\pi}\int_0^{\pi}\int_0^{2}\rho^3\sin^2(\phi)\,dr\,d\phi\,d\theta$$
$$= \int_0^{\pi}\int_0^{\pi}\left[\frac{1}{4}\rho^4\sin^2(\phi)\right]_{\rho=0}^{\rho=2}\,d\phi\,d\theta$$
$$= 4\int_0^{\pi}\int_0^{\pi}\sin^2(\phi)\,d\phi\,d\theta$$
$$= 2\int_0^{\pi}\int_0^{\pi}[1-\cos(2\phi)]\,d\phi\,d\theta$$
$$= 2\int_0^{\pi}\left[\phi-\frac{1}{2}\sin(2\phi)\right]_{\phi=0}^{\phi=\pi}\,d\theta$$
$$= 2\pi\int_0^{\pi}d\theta$$
$$= 2\pi\Big[\theta\Big]_0^{\pi}$$
$$= 2\pi^2.$$

[3] 5. (a) Since a direction vector for the line segment is given by $\langle 2,\ 0,\ 6\rangle$ and $(0,1,-3)$ is a point on the line, the curve C can be parametrised by the function

$$\mathbf{r}(t) = \langle 2t,\ 1,\ -3+6t\rangle$$

for $0 \le t \le 1$. Observe that

$$\mathbf{r}'(t) = \langle 2,\ 0,\ 6\rangle$$

and

$$\mathbf{F}(\mathbf{r}(t)) = \langle 2(2t)-1,\ -(-3+6t),\ 2t+3+(-3+6t)\rangle = \langle 4t-1,\ 3-6t,\ 8t\rangle.$$

Thus

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = (4t-1)(2)+(3-6t)(0)+(8t)(6) = 56t-2$$

and so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (56t-2)\,dt$$
$$= \Big[28t^2-2t\Big]_0^1$$
$$= 26.$$

[5] (b) We have

$$\nabla f = \langle y^2,\ 2xy\rangle.$$

The curve C can be parametrised by the function

$$\mathbf{r}(t) = \langle 2\cos(t),\ 2\sin(t)\rangle$$

for $0 \le t \le \frac{\pi}{2}$ so

$$\mathbf{r}'(t) = \langle -2\sin(t),\ 2\cos(t)\rangle.$$

Hence

$$\nabla f = \langle 4\sin^2(t),\ 8\cos(t)\sin(t)\rangle$$

and

$$\nabla f \cdot \mathbf{r}'(t) = 4\sin^2(t)[-2\sin(t)] + 8\cos(t)\sin(t)[2\cos(t)] = -8\sin^3(t) + 16\cos^2(t)\sin(t).$$

The line integral can then be written

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{s} &= \int_0^{\frac{\pi}{2}} [-8\sin^3(t) + 16\cos^2(t)\sin(t)]\,dt \\
&= \int_0^{\frac{\pi}{2}} [-8\sin(t)(1-\cos^2(t)) + 16\cos^2(t)\sin(t)]\,dt \\
&= \int_0^{\frac{\pi}{2}} [-8\sin(t) + 24\cos^2(t)\sin(t)]\,dt \\
&= [8\cos(t) - 8\cos^3(t)]_0^{\frac{\pi}{2}} \\
&= 0.
\end{aligned}$$

[4] 6. (a) The projection of S onto the xy-plane is the line $2x + y = 6$ or $y = 6 - 2x$, and because we are interested only in the first octant, $x = 0$ and $y = 0$ are also boundary curves. Thus the region of integration is defined by $0 \le y \le 6 - 2x$ and $0 \le x \le 3$. Since S is the graph of $z = 6 - 2x - y$, a normal to S is given by the vector $\langle 2,\ 1,\ 1\rangle$. Hence

$$\mathbf{F} \cdot \langle 2,\ 1,\ 1\rangle = 4y - xy$$

and so

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^{6-2x} (4y - xy)\,dy\,dx \\
&= \int_0^3 \left[2y^2 - \frac{1}{2}xy^2\right]_{y=0}^{y=6-2x} dx \\
&= \int_0^3 (-2x^3 + 20x^2 - 66x + 72)\,dx \\
&= \left[-\frac{1}{2}x^4 + \frac{20}{3}x^3 - 33x^2 + 72x\right]_0^3 \\
&= \frac{117}{2}.
\end{aligned}$$

[5] (b) We will parametrise S using cylindrical coordinates. The equation of the cylinder becomes

$$r^2\cos^2(\theta) + r^2\sin^2(\theta) = 4 \quad\Longrightarrow\quad r^2 = 4 \quad\Longrightarrow\quad r = 2.$$

Thus an appropriate parametrisation is via the function

$$\mathbf{R}(\theta, z) = \langle 2\cos(\theta),\ 2\sin(\theta),\ z\rangle$$

where $0 \le \theta \le 2\pi$ and $0 \le z \le 3$. Now we have

$$\mathbf{R}_\theta = \langle -2\sin(\theta),\ 2\cos(\theta),\ 0\rangle$$

and

$$\mathbf{R}_z = \langle 0,\ 0,\ 1\rangle$$

so

$$\mathbf{R}_\theta \times \mathbf{R}_z = \langle 2\cos(\theta),\ 2\sin(\theta),\ 0\rangle.$$

We can see that this is oriented outward because, for instance, when $\theta = 0$ — which would be true of any point $(2, 0, z)$ on the cylinder — $\mathbf{R}_\theta \times \mathbf{R}_z$ becomes the vector $\langle 2,\ 0,\ 0\rangle$ which points away from the origin. Since

$$\mathbf{F} = \langle 2\cos(\theta), 2\sin(\theta), e^z\rangle$$

we have

$$\mathbf{F}\cdot(\mathbf{R}_\theta \times \mathbf{R}_z) = 4\cos^2(\theta) + 4\sin^2(\theta) + 0 = 4$$

and so

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^3 4\,dz\,d\theta \\
&= \int_0^{2\pi}\Big[4z\Big]_{z=0}^{z=3} d\theta \\
&= 12\int_0^{2\pi} d\theta \\
&= 12\Big[\theta\Big]_0^{2\pi} \\
&= 24\pi.
\end{aligned}$$