MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 6	**MATH 3202**	Spring 2019

SOLUTIONS

[5] 1. (a) The surface is the graph of the function $z = 2x^2 + 8y + 3 = f(x, y)$. Hence

$$f_x(x, y) = 4x \quad \text{and} \quad f_y(x, y) = 8$$

so

$$\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} = \sqrt{16x^2 + 64 + 1} = \sqrt{16x^2 + 65}.$$

The domain of integration, described by the indicated triangle, is bounded by the lines $y = 0$, $x = 1$ and $y = 8x$. Hence it is defined by $0 \le y \le 8x$ and $0 \le x \le 1$. Thus

$$\begin{aligned} A &= \int_0^1 \int_0^{8x} \sqrt{16x^2 + 65}\, dy\, dx \\ &= \int_0^1 \Big[y\sqrt{16x^2 + 65} \Big]_{y=0}^{y=8x} dx \\ &= 8 \int_0^1 x\sqrt{16x^2 + 65}\, dx. \end{aligned}$$

We let $u = 16x^2 + 65$ so $\frac{1}{32}\, du = x\, dx$. When $x = 0$, $u = 65$ and when $x = 1$, $u = 81$. The integral becomes

$$\begin{aligned} A &= \frac{1}{4} \int_{65}^{81} \sqrt{u}\, du \\ &= \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_{65}^{81} \\ &= \frac{1}{6}(729 - 65^{\frac{3}{2}}). \end{aligned}$$

[6] (b) We have $f(x, y) = x^2 + y^2$ so

$$f_x(x, y) = 2x \quad \text{and} \quad f_y(x, y) = 2y$$

so

$$\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}.$$

The domain of integration is determined by the projection of the cylinder onto the xy-plane, namely the circle $x^2 + y^2 = 2$. This suggests that we should use polar coordinates, for which the circle has the equation $r = \sqrt{2}$. Thus $0 \le r \le \sqrt{2}$ and $0 \le \theta \le 2\pi$. Furthermore,

$$\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} = \sqrt{4r^2 \cos^2(\theta) + 4r^2 \sin^2(\theta) + 1} = \sqrt{4r^2 + 1}.$$

Recalling that $dA = r\,dr\,d\theta$ in polar coordinates, we have

$$A = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2+1}\,r\,dr\,d\theta.$$

We let $u = 4r^2 + 1$ so $\frac{1}{8}\,du = r\,dr$. When $r = 0$, $u = 1$ and when $r = \sqrt{2}$, $u = 9$. Thus the integral becomes

$$\begin{aligned} A &= \frac{1}{8} \int_0^{2\pi} \int_1^9 \sqrt{u}\,du\,d\theta \\ &= \frac{1}{8} \int_0^{2\pi} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_{u=1}^{u=9} d\theta \\ &= \frac{13}{6} \int_0^{2\pi} d\theta \\ &= \frac{13}{6} \Big[\theta \Big]_0^{2\pi} \\ &= \frac{13\pi}{3}. \end{aligned}$$

[6] (c) We have

$$\mathbf{R}_u(u,v) = \langle \cos(v),\ \sin(v),\ 0 \rangle \quad \text{and} \quad \mathbf{R}_v(u,v) = \langle -u\sin(v),\ u\cos(v),\ 1 \rangle.$$

Thus

$$\mathbf{R}_u \times \mathbf{R}_v = \langle \sin(v),\ -\cos(v),\ u \rangle$$

and so

$$\|\mathbf{R}_u \times \mathbf{R}_v\| = \sqrt{\sin^2(v) + \cos^2(v) + u^2} = \sqrt{1+u^2}.$$

Hence

$$\begin{aligned} A &= \int_0^1 \int_0^{\pi} \sqrt{1+u^2}\,dv\,du \\ &= \int_0^1 \left[v\sqrt{1+u^2} \right]_{v=0}^{v=\pi} du \\ &= \pi \int_0^1 \sqrt{1+u^2}\,du. \end{aligned}$$

We let $u = \tan(\theta)$ so $du = \sec^2(\theta)\,d\theta$. Then

$$\sqrt{1+u^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = \sec(\theta).$$

When $u = 0$, $\theta = 0$ and when $u = 1$, $\theta = \frac{\pi}{4}$. Hence the integral becomes

$$\begin{aligned} A &= \pi \int_0^{\frac{\pi}{4}} \sec(\theta)\sec^2(\theta)\,d\theta \\ &= \pi \int_0^{\frac{\pi}{4}} \sec^3(\theta)\,d\theta. \end{aligned}$$

Using integration by parts, we find that

$$\begin{aligned} A &= \frac{\pi}{2}\left[\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right]. \end{aligned}$$

[5] 2. (a) Since $z = 2x + 2y - 4 = f(x, y)$, we have

$$f_x(x,y) = f_y(x,y) = 2 \quad \text{and} \quad \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1} = \sqrt{4+4+1} = 3.$$

Furthermore, the integrand becomes

$$yz = y(2x + 2y - 4) = 2xy + 2y^2 - 4y.$$

The projection of the indicated plane onto the xy-plane is the line with equation $2x+2y = 4$ so $y = 2 - x$. Since we are interested only in the part of the plane in the first octant, $x = 0$ and $y = 0$ also bound this region. Hence it is defined by $0 \le y \le 2 - x$ and $0 \le x \le 2$, and so the surface integral can be written

$$\begin{aligned} \iint_S xz\,dS &= \int_0^2 \int_0^{2-x} (2xy + 2y^2 - 4y)\cdot 3\,dy\,dx \\ &= 3\int_0^2 \left[xy^2 + \frac{2}{3}y^3 - 2y^2\right]_{y=0}^{y=2-x} dx \\ &= \int_0^2 (x^3 - 6x^2 + 12x - 8)\,dx \\ &= \left[\frac{1}{4}x^4 - 2x^3 + 6x^2 - 8x\right]_0^2 \\ &= -4. \end{aligned}$$

[4] (b) We have already found that

$$\|\mathbf{R}_u(u,v) \times \mathbf{R}_v(u,v)\| = \sqrt{1+u^2}.$$

Furthermore, the integrand can be written $xz = uv\cos(v)$. Hence

$$\iint_S yz\,dS = \int_0^1 \int_0^{\pi} uv\sin(v)\sqrt{1+u^2}\,dv\,du.$$

The integral with respect to v can be evaluated by parts, giving

$$\iint_S xz\,dS = \pi\int_0^1 u\sqrt{1+u^2}\,du.$$

Now let $w = 1 + u^2$ so $\frac{1}{2}\,dw = u\,du$. When $u = 0$, $w = 1$ and when $u = 1$, $w = 2$. The integral becomes

$$\iint\limits_S xz\,dS = \frac{\pi}{2}\int_1^2 \sqrt{w}\,dw$$
$$= \frac{\pi}{2}\left[\frac{2}{3}w^{\frac{3}{2}}\right]_1^2$$
$$= \frac{\pi}{3}(2\sqrt{2} - 1).$$

[5] 3. (a) In the xy-plane, the plane $x + y + z = 1$ becomes the line $x + y = 1$, while the plane $x + 2y + z = 1$ becomes the line $x + 2y = 1$. Furthermore, $x = 0$ and $y = 0$ are boundary curves because we are only interested in the first octant. The projection of E in the xy-plane is then most easily viewed as a Type 1 region (that is, with boundary curves that are functions of x) so we can rewrite the lines as $y = 1 - x$ and $y = \frac{1}{2} - \frac{1}{2}x$. Then the projection is bounded by $\frac{1}{2} - \frac{1}{2}x \le y \le 1 - x$ and $0 \le x \le 1$. Furthermore, since E itself is bounded by the surfaces $z = 1 - x - y$ and $z = 1 - x - 2y$, it is defined by $1 - x - 2y \le z \le 1 - x - y$. Thus

$$V = \iiint\limits_E dV = \int_0^1 \int_{\frac{1}{2}-\frac{1}{2}x}^{1-x} \int_{1-x-2y}^{1-x-y} dz\,dy\,dx$$
$$= \int_0^1 \int_{\frac{1}{2}-\frac{1}{2}x}^{1-x} \Big[z\Big]_{z=1-x-2y}^{z=1-x-y} dy\,dx$$
$$= \int_0^1 \int_{\frac{1}{2}-\frac{1}{2}x}^{1-x} y\,dy\,dx$$
$$= \int_0^1 \left[\frac{1}{2}y^2\right]_{y=\frac{1}{2}-\frac{1}{2}x}^{y=1-x} dx$$
$$= \frac{1}{8}\int_0^1 (3x^2 - 6x + 3)\,dx$$
$$= \frac{1}{8}\Big[x^3 - 3x^2 + 3x\Big]_0^1$$
$$= \frac{1}{8}.$$

[4] (b) We can set up the iterated version of this triple integral exactly as in part (a), since the

change of integrand does not affect the geometry of the problem. Thus

$$\iiint\limits_E (x+y)\,dV = \int_0^1 \int_{\frac{1}{2}-\frac{1}{2}x}^{1-x} \int_{1-x-2y}^{1-x-y} (x+y)\,dz\,dy\,dx$$

$$= \int_0^1 \int_{\frac{1}{2}-\frac{1}{2}x}^{1-x} \Big[(x+y)z\Big]_{z=1-x-2y}^{z=1-x-y} dy\,dx$$

$$= \int_0^1 \int_{\frac{1}{2}-\frac{1}{2}x}^{1-x} (xy+y^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}x^2y + xy^2\right]_{x=\frac{1}{2}-\frac{1}{2}x}^{x=1-x} dx$$

$$= \frac{1}{24}\int_0^1 (2x^3+3x^2-12x+7)\,dx$$

$$= \frac{1}{24}\left[\frac{1}{2}x^4 + x^3 - 6x^2 + 7x\right]_0^1$$

$$= \frac{5}{48}.$$

[5] 4. The projection of E onto the xy-plane consists of the unit circle, which in Cartesian coordinates can be defined by $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ and $-1 \le x \le 1$. (Alternatively, we could also use $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$ and $-1 \le y \le 1$.) Since E itself is bounded by $0 \le z \le y$ we have

$$\iiint\limits_E (x+y)z\,dV = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^y (x+y)z\,dz\,dy\,dx$$

$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\frac{1}{2}(x+y)z^2\right]_{z=0}^{z=y} dy\,dx$$

$$= \frac{1}{2}\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (xy^2+y^3)\,dy\,dx$$

$$= \frac{1}{2}\int_{-1}^1 \left[\frac{1}{3}xy^3 + \frac{1}{4}y^4\right]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

$$= \frac{1}{3}\int_{-1}^1 x(1-x^2)^{\frac{3}{2}}\,dx.$$

Let $u = 1-x^2$ so $-\frac{1}{2}\,du = x\,dx$. When $x=-1$, $u=0$ and when $x=1$, $u=0$. Since the bounds of integration are now the same, we immediately have

$$\iiint\limits_E (x+y)z\,dV = 0.$$