MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 5 | **MATH 3202** | SPRING 2019

SOLUTIONS

[4] 1. (a) We have

$$\nabla f(x,y) = \langle 2xy^3,\ 3x^2y^2\rangle \quad \text{and} \quad \nabla f(2,-1) = \langle -4,\ 12\rangle.$$

A unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle -\frac{3}{5},\ \frac{4}{5}\right\rangle.$$

Hence

$$\begin{aligned} D_{\mathbf{u}}f(2,-1) &= \nabla f(2,-1)\cdot\mathbf{u} \\ &= \langle -4,\ 12\rangle \cdot \left\langle -\frac{3}{5},\ \frac{4}{5}\right\rangle \\ &= \frac{12}{5} + \frac{48}{5} \\ &= 12. \end{aligned}$$

[4] (b) We have

$$\nabla f(x,y,z) = \langle e^{-yz},\ -xze^{-yz},\ -xye^{-yz}\rangle \quad \Longrightarrow \quad \nabla f(1,0,-3) = \langle 1,\ 3,\ 0\rangle.$$

A unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right\rangle = \left\langle \frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{3}\right\rangle.$$

Hence

$$\begin{aligned} D_{\mathbf{u}}f(1,0,-3) &= \nabla f(1,0,-3)\cdot\mathbf{u} \\ &= \langle 1,\ 3,\ 0\rangle \cdot \left\langle \frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{3}\right\rangle \\ &= \frac{\sqrt{6}}{6} - \frac{\sqrt{6}}{2} \\ &= -\frac{\sqrt{6}}{3}. \end{aligned}$$

[3] 2. (a) We have

$$\nabla T(x,y,z) = \left\langle \frac{\cos(x)\cos(y)}{z^2+1},\ \frac{-\sin(x)\sin(y)}{z^2+1},\ -\frac{2z\sin(x)\cos(y)}{(z^2+1)^2} \right\rangle$$

so

$$\nabla T(0,0,0) = \langle 1,\ 0,\ 0\rangle = \mathbf{i}.$$

Hence this is the direction in which the maximum rate of change occurs, and its magnitude is

$$\|\nabla T(0,0,0)\| = 1,$$

that is, the temperature increases by at most 1°C per metre.

[2] (b) The drone's direction is described by the vector $\mathbf{v} = \langle 10,\ 5,\ 10\rangle$ with corresponding unit vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{2}{3},\ \frac{1}{3},\ \frac{2}{3} \right\rangle.$$

Hence the rate of change of the temperature in this direction is given by

$$\begin{aligned} D_{\mathbf{u}}T(0,0,0) &= \nabla T(0,0,0)\cdot\mathbf{u} \\ &= \langle 1,\ 0,\ 0\rangle \cdot \left\langle \frac{2}{3},\ \frac{1}{3},\ \frac{2}{3} \right\rangle \\ &= \frac{2}{3}, \end{aligned}$$

that is, the temperature increases at a rate of $\frac{2}{3}$°C per metre.

[6] 3. We have

$$\nabla f(x,y) = \langle 2x + y\cos(xy),\ x\cos(xy)\rangle \quad\Longrightarrow\quad \nabla f(1,0) = \langle 2,\ 1\rangle.$$

Given an arbitrary unit vector $\mathbf{u} = \langle a,\ b\rangle$ we then have

$$\begin{aligned} D_{\mathbf{u}}f(1,0) &= \nabla f(1,0)\cdot\mathbf{u} \\ &= \langle 2,\ 1\rangle\cdot\langle a,\ b\rangle \\ &= 2a+b. \end{aligned}$$

We set $2a+b=0$ so then $b=-2a$. Thus any such vector will have the form

$$\mathbf{u} = \langle a,\ -2a\rangle.$$

This will be a unit vector if

$$a^2 + (-2a)^2 = 1 \quad\Longrightarrow\quad 5a^2 = 1 \quad\Longrightarrow\quad a = \pm\frac{\sqrt{5}}{5}.$$

Thus the two desired unit vectors are

$$\left\langle \frac{\sqrt{5}}{5},\ -\frac{2\sqrt{5}}{5} \right\rangle \quad\text{and}\quad \left\langle -\frac{\sqrt{5}}{5},\ \frac{2\sqrt{5}}{5} \right\rangle.$$

[4] 4. On the lefthand side, we have

$$\nabla(fg) = \left\langle \frac{d}{dx}[fg],\ \frac{d}{dy}[fg] \right\rangle = \langle f_x g + g_x f,\ f_y g + g_y f \rangle.$$

On the righthand side, we have

$$\begin{aligned} f\nabla g + g\nabla f &= f\langle g_x,\ g_y\rangle + g\langle f_x,\ f_y\rangle \\ &= \langle fg_x,\ fg_y\rangle + \langle gf_x,\ gf_y\rangle \\ &= \langle fg_x + gf_x,\ fg_y + gf_y\rangle \\ &= \langle f_x g + g_x f,\ f_y g + g_y f\rangle \\ &= \nabla(fg) \end{aligned}$$

as required.

[4] 5. (a) We can treat the ellipsoid as a level surface $f(x,y,z) = 1$ where

$$f(x,y,z) = 5x^2 + y^2 + 3z^2.$$

Thus

$$\nabla f(x,y,z) = \langle 10x,\ 2y,\ 6z\rangle$$

and so a normal to the tangent plane is

$$\mathbf{n} = \nabla f(1,4,-1) = \langle 10,\ 8,\ -6\rangle.$$

The equation of the tangent plane is therefore

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{x} - \mathbf{x}_0) &= 0 \\ \langle 10,\ 8,\ -6\rangle \cdot \langle x-1,\ y-4,\ z+1\rangle &= 0 \\ 10(x-1) + 8(y-4) - 6(z+1) &= 0 \\ 10x + 8y - 6z &= 48 \\ 5x + 4y - 3z &= 24. \end{aligned}$$

[2] (b) The normal vector $\mathbf{n} = \langle 10,\ 8,\ -6\rangle$ to the tangent plane will also serve as a direction vector for the normal line. Thus its equation is

$$\langle x,\ y,\ z\rangle = \langle 1,\ 4,\ -1\rangle + t\langle 10,\ 8,\ -6\rangle = \langle 1+10t,\ 4+8t,\ -1-6t\rangle.$$

[6] 6. We can treat the hyperbolic paraboloid as a level surface $f(x,y,z) = 0$ where

$$f(x,y,z) = \frac{x^2}{4} - \frac{y^2}{3} - z.$$

Thus

$$\nabla f(x,y,z) = \left\langle \frac{1}{2}x,\ -\frac{2}{3}y,\ -1 \right\rangle$$

and so a normal vector to the tangent plane at the point $P(x_0, y_0, z_0)$ will be

$$\mathbf{n} = \left\langle \frac{1}{2}x_0,\ -\frac{2}{3}y_0,\ -1 \right\rangle.$$

In order for a tangent plane to be parallel to the indicated plane, its normal vector must be parallel to the vector $\langle 1,\ 4,\ 6\rangle$. Thus there must exist a constant k for which

$$\frac{1}{2}x_0 = k, \quad -\frac{2}{3}y_0 = 4k, \quad -1 = 6k.$$

From the latter equation $k = -\frac{1}{6}$ and therefore $x_0 = -\frac{1}{3}$ and $y_0 = 1$. Since P must lie on the hyperbolic paraboloid, then,

$$z_0 = \frac{\left(-\frac{1}{3}\right)^2}{4} - \frac{1^2}{3} = -\frac{11}{36}.$$

Therefore the equation of the tangent plane must be

$$\begin{aligned}
\left\langle -\frac{1}{6},\ -\frac{2}{3},\ -1 \right\rangle \cdot \left\langle x + \frac{1}{3},\ y - 1,\ z + \frac{11}{36} \right\rangle &= 0 \\
-\frac{1}{6}x - \frac{1}{18} - \frac{2}{3}y + \frac{2}{3} - z - \frac{11}{36} &= 0 \\
-\frac{1}{6}x - \frac{2}{3}y - z &= -\frac{11}{36} \\
x + 4y + 6z &= \frac{11}{6} \\
6x + 24y + 36z &= 11.
\end{aligned}$$

[5] 7. We can treat the sphere as a level surface $f(x, y, z) = R^2$ where

$$f(x, y, z) = (x - a)^2 + (y - b)^2 + (z - c)^2.$$

Hence

$$\nabla f(x, y, z) = \langle 2(x - a),\ 2(y - b),\ 2(z - c)\rangle.$$

At any point $P(x_0, y_0, z_0)$, a direction vector for the normal line will be

$$\nabla f(x_0, y_0, z_0) = \langle 2x_0 - 2a,\ 2y_0 - 2b,\ 2z_0 - 2c\rangle$$

and therefore the equation of the normal line will be

$$\begin{aligned}
\langle x,\ y,\ z\rangle &= \langle x_0,\ y_0,\ z_0\rangle + t\langle 2x_0 - 2a,\ 2y_0 - 2b,\ 2z_0 - 2c\rangle \\
&= \langle x_0 + 2tx_0 - 2at,\ y_0 + 2ty_0 - 2bt,\ z_0 + 2tz_0 - 2ct\rangle \\
&= \langle (2t + 1)x_0 - 2at,\ (2t + 1)y_0 - 2bt,\ (2t + 1)z_0 - 2ct\rangle.
\end{aligned}$$

The centre of the sphere is the point (a, b, c) and we can see that the point lies on the normal line when $t = -\frac{1}{2}$. Hence the centre of the sphere lies on every normal line.