MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 4	**MATH 3202**	SPRING 2019

SOLUTIONS

[2] 1. (a) The velocity vector function is

$$\mathbf{v}(t) = \langle e^t\sin(t) + e^t\cos(t),\ e^t\cos(t) - e^t\sin(t),\ 0\rangle.$$

[2] (b) The speed function is

$$\begin{aligned} v(t) = \|\mathbf{v}(t)\| &= \sqrt{[e^t\sin(t)+e^t\cos(t)]^2 + [e^t\cos(t)-e^t\sin(t)]^2 + 0^2} \\ &= \sqrt{2e^{2t}\sin^2(t) + 2e^{2t}\cos^2(t)} \\ &= \sqrt{2e^{2t}} \\ &= \sqrt{2}e^t. \end{aligned}$$

[2] (c) The acceleration vector function is

$$\begin{aligned} \mathbf{a}(t) &= \langle e^t\sin(t) + e^t\cos(t) + e^t\cos(t) - e^t\sin(t), \\ &\qquad e^t\cos(t) - e^t\sin(t) - e^t\sin(t) - e^t\cos(t),\ 0\rangle \\ &= \langle 2e^t\cos(t),\ -2e^t\sin(t),\ 0\rangle. \end{aligned}$$

[2] (d) The tangential component of the acceleration is given by

$$\begin{aligned} a_T(t) = \frac{\mathbf{a}(t)\cdot\mathbf{v}(t)}{\|\mathbf{v}(t)\|} &= \frac{-2e^{2t}\cos(t)\sin(t) + 2e^{2t}\sin^2(t) + 2e^{2t}\cos^2(t) + 2e^{2t}\cos(t)\sin(t)}{\sqrt{2}e^t} \\ &= \frac{2e^{2t}}{\sqrt{2}e^t} \\ &= \sqrt{2}e^t. \end{aligned}$$

[2] (e) We have

$$\mathbf{a}\times\mathbf{v} = \langle 0,\ 0,\ -2e^{2t}\rangle.$$

Thus the normal component of the acceleration is given by

$$\begin{aligned} a_N(t) = \frac{\|\mathbf{a}(t)\times\mathbf{v}(t)\|}{\|\mathbf{v}(t)\|} &= \frac{2e^{2t}}{\sqrt{2}e^t} \\ &= \sqrt{2}e^t. \end{aligned}$$

Thus we have the unusual case that $a_T(t) = a_N(t)$.

[7] 2. We suppose that Peter is at a point with coordinates $(x, y) = (0, 5)$ so that $\mathbf{r}(0) = \langle 0,\ 5\rangle$. Since the angle of the shot is $30°$, a corresponding unit vector is $\langle \cos(30°),\ \sin(30°)\rangle = \langle \frac{\sqrt{3}}{2},\ \frac{1}{2}\rangle$. Thus, since the initial speed of the paintball is 20 m/sec, we have that

$$\mathbf{v}(0) = 20\left\langle \frac{\sqrt{3}}{2},\ \frac{1}{2}\right\rangle = \langle 10\sqrt{3},\ 10\rangle.$$

We want to find the x-coordinate of the point where the paintball lands, and determine whether this is less than, greater than, or equal to 50.

Assuming that gravity is the only force acting on the paintball, we have

$$\mathbf{a}(t) = \langle 0,\ -9.8\rangle.$$

Thus

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle C_1,\ -9.8t + C_2\rangle.$$

Substituting $t = 0$ we get

$$\mathbf{v}(0) = \langle C_1,\ C_2\rangle$$

and so $C_1 = 10\sqrt{3}$ and $C_2 = 10$. Hence

$$\mathbf{v}(t) = \langle 10\sqrt{3},\ -9.8t + 10\rangle.$$

Next,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \langle 10\sqrt{3}t + D_1,\ -4.9t^2 + 10t + D_2\rangle.$$

Again substituting $t = 0$ we obtain

$$\mathbf{r}(0) = \langle D_1,\ D_2\rangle$$

so $D_1 = 0$ and $D_2 = 5$. Now we know that

$$\mathbf{r}(t) = \langle 10\sqrt{3}t,\ -4.9t^2 + 10t + 5\rangle.$$

Finally, we need to determine when the paintball will strike the ground — that is, when $y = 0$. Thus we set

$$-4.9t^2 + 10t + 5 = 0$$

and find that $t \approx -0.42$ or $t \approx 2.46$. By convention, we neglect the negative option and find that, in the latter case,

$$x \approx 10\sqrt{3}(2.46) = 42.54.$$

Hence the paintball travels only about 42.5 metres along the ground, making this an <u>undershoot</u> and missing Jodie.

[3] 3. Since the two planes are parallel, they must share the same normal $\langle 9,\ 2,\ 6\rangle$. Hence the equation of the desired plane is given by

$$\begin{aligned}\langle 9,\ 2,\ 6\rangle \cdot \langle x-2,\ y-0,\ z+1\rangle &= 0\\ 9(x-2)+2y+6(z+1) &= 0\\ 9x+2y+6z &= 12.\end{aligned}$$

[3] 4. The $x=0$ trace is

$$-4y+9z^2=1 \quad\Longrightarrow\quad y=\frac{9}{4}z^2-\frac{1}{4},$$

which is a <u>parabola</u>.

The $y=0$ trace is

$$x^2+9z^2=1$$

which is an <u>ellipse</u>.

The $z=0$ trace is

$$x^2-4y=1 \quad\Longrightarrow\quad y=\frac{1}{4}x^2-\frac{1}{4},$$

which is also a <u>parabola</u>.

[2] 5. (a) The standard form of an ellipsoid centred at $(0,0,0)$ is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

so this is an ellipsoid with $a=\sqrt{5}$, $b=\sqrt{6}$ and $c=\sqrt{7}$.

[2] (b) We can rearrange the given expression as

$$z^2=1-x^2-3y^2 \quad\Longrightarrow\quad x^2+3y^2+z^2=1,$$

so this is a semi-ellipsoid (since $z\geq 0$ only) with $a=c=1$ and $b=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$.

[3] (c) This is not an ellipsoid, but rather an <u>elliptic paraboloid</u> with vertex at $(0,0)$, of the form

$$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

for $a=1$ and $b=\frac{\sqrt{3}}{3}$. (We might also note that, unlike an ellipsoid for which all the traces are ellipses, in this curve only the $z=k$ traces are ellipses. The $x=k$ and $y=k$ traces are parabolas.)

[3] (d) Since this expression has both quadratic and linear terms, we should complete the square as necessary. First,

$$6x^2+48x=6(x^2+8x)=6(x+4)^2-96.$$

There is no linear term in y, so we next consider

$$4z^2 - 4z = 4(z^2 - z) = 4\left(z - \frac{1}{2}\right)^2 - 1.$$

Thus this expression can be rewritten

$$\begin{aligned} [6(x+4)^2 - 96] + 12y^2 + \left[4\left(z - \frac{1}{2}\right)^2 - 1\right] + 85 = 0 \\ 6(x+4)^2 + 12y^2 + 4\left(z - \frac{1}{2}\right)^2 = 12 \\ \frac{(x+4)^2}{2} + y^2 + \frac{\left(z - \frac{1}{2}\right)^2}{3} = 1. \end{aligned}$$

This is the equation of an ellipsoid centred at $\left(-4, 0, \frac{1}{2}\right)$ with $a = \sqrt{2}$, $b = 1$ and $c = \sqrt{3}$.

[3] (e) This is a surface with parametric equations

$$x = 3\cos(u)\sin(v), \quad y = \frac{1}{2}\sin(u)\sin(v), \quad z = 2\cos(v).$$

Then observe that

$$\begin{aligned} \left(\frac{x}{3}\right)^2 + (2y)^2 + \left(\frac{z}{2}\right)^2 &= \cos^2(u)\sin^2(v) + \sin^2(u)\sin^2(v) + \cos^2(v) \\ &= \sin^2(v) + \cos^2(v) \\ &= 1. \end{aligned}$$

Hence this is an ellipsoid with $a = 3$, $b = \frac{1}{2}$ and $c = 2$.

[4] 6. First note that, at the given point,

$$z = f(3,1) = 9 - \frac{1}{2} = \frac{17}{2}.$$

Next we have

$$f_x(x,y) = 2x\sqrt{y} \quad \Longrightarrow \quad f_x(3,1) = 6$$

and

$$f_y(x,y) = \frac{x^2}{2\sqrt{y}} + \frac{5}{2y^6} \quad \Longrightarrow \quad f_y(3,1) = \frac{9}{2} + \frac{5}{2} = 7.$$

Thus a normal to the tangent plane is $\mathbf{n} = \langle 6,\ 7,\ -1\rangle$ and so an equation of the tangent plane is

$$\begin{aligned} \langle 6,\ 7,\ -1\rangle \cdot \left\langle x - 3,\ y - 1,\ z - \frac{17}{2}\right\rangle = 0 \\ 6x - 18 + 7y - 7 - z + \frac{17}{2} = 0 \\ 6x + 7y - z = \frac{33}{2} \\ 12x + 14y - 2z = 33. \end{aligned}$$