MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 3 **MATH 3202** SPRING 2019

SOLUTIONS

[5] 1. The area is given by

$$A = \int_C x^{-1} y \, ds$$

where the integrand becomes

$$x^{-1}y = (t^3)^{-1} \left(\frac{1}{4} t^4 \right) = \frac{1}{4} t$$

and

$$\mathbf{r}'(t) = \langle 3t^2, t^3 \rangle \quad \implies \quad \|\mathbf{r}'(t)\| = \sqrt{9t^4 + t^6} = t^2 \sqrt{9 + t^2}.$$

Hence we can write

$$\begin{aligned} A &= \int_3^4 \frac{1}{4} t \cdot t^2 \sqrt{9 + t^2} \, dt \\ &= \frac{1}{4} \int_3^4 t^3 \sqrt{9 + t^2} \, dt. \end{aligned}$$

We let $u = 9 + t^2$ so $\frac{1}{2} \, du = t \, dt$. Furthermore, $t^2 = u - 9$. The integral becomes

$$\begin{aligned} A &= \frac{1}{8} \int_{18}^{25} (u - 9) \sqrt{u} \, du \\ &= \frac{1}{8} \int_{18}^{25} \left(u^{\frac{3}{2}} - 9\sqrt{u} \right) \, du \\ &= \frac{1}{8} \left[\frac{2}{5} u^{\frac{5}{2}} - 6u^{\frac{3}{2}} \right]_{18}^{25} \\ &= \frac{125}{2} - \frac{81\sqrt{2}}{10}. \end{aligned}$$

[8] 2. Let the first part of the curve be C_1 and be parametrised by $\mathbf{r}_1(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le \frac{\pi}{2}$. Observe that

$$\mathbf{r}_1'(t) = \langle -\sin(t), \cos(t) \rangle$$

is continuous and is non-zero for all t so C_1 is smooth. Likewise, let the second part of the curve be C_2 and be parametrised by $\mathbf{r}_2(t) = \langle t, t^2 + 1 \rangle$ for $0 \le t \le 2$. Then

$$\mathbf{r}_2'(t) = \langle 1, 2t \rangle$$

which, again, is continuous and non-zero for all t. Hence C_2 is also smooth. Since C is the union of two smooth curves, it is piecewise smooth and we can write

$$\int_C xy\,ds = \int_{C_1} x\,ds + \int_{C_2} x\,ds.$$

For the first integral, then, we have

$$x = \cos(t) \quad \text{and} \quad \|\mathbf{r}_1'(t)\| = \sqrt{\sin^2(t) + \cos^2(t)} = 1.$$

Thus

$$\begin{aligned}\int_{C_1} x\,ds &= \int_0^{\frac{\pi}{2}} \cos(t)\cdot 1\,dt\\ &= \int_0^{\frac{\pi}{2}} \cos(t)\,dt\\ &= \Big[\sin(t)\Big]_0^{\frac{\pi}{2}}\\ &= 1.\end{aligned}$$

For the second integral, we have

$$x = t \quad \text{and} \quad \|\mathbf{r}_2'(t)\| = \sqrt{1+4t^2}.$$

Hence

$$\int_{C_2} x\,ds = \int_0^2 t\sqrt{1+4t^2}\,dt.$$

Let $u = 1 + 4t^2$ so $\frac{1}{8}\,du = t\,dt$. The integral becomes

$$\begin{aligned}\int_{C_2} x\,ds &= \frac{1}{8}\int_1^{17} \sqrt{u}\,du\\ &= \frac{1}{8}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{17}\\ &= \frac{17\sqrt{17}}{12} - \frac{1}{12}.\end{aligned}$$

Finally, then,

$$\int_C x\,ds = 1 + \frac{17\sqrt{17}}{12} - \frac{1}{12} = \frac{17\sqrt{17}}{12} + \frac{11}{12}.$$

[5] 3. (a) One way to parametrise the line segment is

$$\mathbf{r}(t) = \langle 2, -3, 0\rangle + t(\langle 8, 0, -2\rangle - \langle 2, -3, 0\rangle) = \langle 2+6t, -3+3t, -2t\rangle$$

for $0 \leq t \leq 1$. Hence

$$x - yz = (2+6t) - (-3+3t)(-2t) = 6t^2 + 2$$

and

$$\mathbf{r}'(t) = \langle 6, 3, -2 \rangle \quad \Longrightarrow \quad \|\mathbf{r}'(t)\| = \sqrt{36+9+4} = 7.$$

Now we can write

$$\begin{aligned}\int_C (x - yz)\, ds &= \int_0^1 (6t^2+2) \cdot 7\, dt \\ &= 7\Big[2t^3 + 2t\Big]_0^1 \\ &= 28.\end{aligned}$$

[5] (b) We have

$$x - yz = \sin(2t) - 2t\cos(2t)$$

and

$$\mathbf{r}'(t) = \langle 2\cos(2t), 2, -2\sin(2t) \rangle \quad \Longrightarrow \quad \|\mathbf{r}'(t)\| = \sqrt{4\cos^2(2t) + 4 + 4\sin^2(2t)} = 2\sqrt{2}.$$

Hence

$$\begin{aligned}\int_C (x - yz)\, ds &= \int_0^{\frac{\pi}{4}} [\sin(2t) - 2t\cos(2t)] \cdot 2\sqrt{2}\, dt \\ &= 2\sqrt{2}\left[-\frac{1}{2}\cos(2t) - t\sin(2t) - \frac{1}{2}\cos(2t)\right]_0^{\frac{\pi}{4}},\end{aligned}$$

where the integral of the second term can be evaluated using integration by parts. Now we have

$$\begin{aligned}\int_C (x - yz)\, ds &= 2\sqrt{2}\Big[-\cos(2t) - t\sin(2t)\Big]_0^{\frac{\pi}{4}} \\ &= 2\sqrt{2} - \frac{\pi\sqrt{2}}{2}.\end{aligned}$$

[2] 4. (a) We have

$$\mathbf{r}'(t) = \langle 3\cos(t), -5\sin(t), -4\cos(t) \rangle$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{9\cos^2(t) + 25\sin^2(t) + 16\cos^2(t)} = 5.$$

Hence

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle \frac{3}{5}\cos(t), -\sin(t), -\frac{4}{5}\cos(t) \right\rangle.$$

[2] (b) From part (a) we find

$$\mathbf{T}'(t) = \left\langle -\frac{3}{5}\sin(t), -\cos(t), \frac{4}{5}\sin(t) \right\rangle$$

so

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{9}{25}\sin^2(t) + \cos^2(t) + \frac{16}{25}\sin^2(t)} = 1.$$

Thus

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{5}.$$

[2] (c) From part (b), we have

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \left\langle -\frac{3}{5}\sin(t), -\cos(t), \frac{4}{5}\sin(t) \right\rangle.$$

[2] (d) From parts (a) and (c),

$$\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5}\cos(t) & -\sin(t) & -\frac{4}{5}\cos(t) \\ -\frac{3}{5}\sin(t) & -\cos(t) & \frac{4}{5}\sin(t) \end{vmatrix} \\
&= \left[-\frac{4}{5}\sin^2(t) - \frac{4}{5}\cos^2(t)\right]\mathbf{i} - \left[\frac{12}{25}\cos(t)\sin(t) - \frac{12}{25}\cos(t)\sin(t)\right]\mathbf{j} \\
&\quad + \left[-\frac{3}{5}\cos^2(t) - \frac{3}{5}\sin^2(t)\right]\mathbf{k} \\
&= \left\langle -\frac{4}{5}, 0, -\frac{3}{5} \right\rangle.
\end{aligned}$$

[2] (e) At $t = 0$,

$$\mathbf{r}(0) = \langle 0, 5, 0 \rangle \quad \text{and} \quad \mathbf{B}(0) = \left\langle -\frac{4}{5}, 0, -\frac{3}{5} \right\rangle.$$

Since $\mathbf{B}(0)$ is a normal to the osculating plane, its equation has the form

$$-\frac{4}{5}x - \frac{3}{5}z = D.$$

And since $(0, 5, 0)$ must be a point in the osculating plane, we see that $D = 0$ so the equation of the plane is

$$-\frac{4}{5}x - \frac{3}{5}z = 0 \quad \Longrightarrow \quad 4x + 3z = 0.$$

[3] 5. We have

$$\mathbf{r}'(t) = \langle 2t, 4\sqrt{2}t, 1\rangle \quad \text{and} \quad \mathbf{r}''(t) = \langle 2, 4\sqrt{2}, 0\rangle.$$

Hence

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 4\sqrt{2}t & 1 \\ 2 & 4\sqrt{2} & 0 \end{vmatrix} = \langle -4\sqrt{2}, 2, 0\rangle$$

and so

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{32+4} = 6.$$

Furthermore,

$$\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 32t^2 + 1} = \sqrt{36t^2+1}.$$

Hence

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{6}{(36t^2+1)^{\frac{3}{2}}}.$$

[4] 6. Since $f'(x) = 3e^{3x}$ and $f''(x) = 9e^{3x}$, we have

$$\begin{aligned} \kappa(x) &= \frac{|f''(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}} \\ &= \frac{9e^{3x}}{[1+9e^{6x}]^{\frac{3}{2}}}. \end{aligned}$$

To find the point of maximum curvature, we compute

$$\kappa'(x) = \frac{27e^{3x}(1-18e^{6x})}{[1+9e^{6x}]^{\frac{5}{2}}}.$$

Setting $\kappa'(x) = 0$ we find

$$1 - 18e^{6x} = 0 \quad \Longrightarrow \quad e^{6x} = \frac{1}{18} \quad \Longrightarrow \quad x = \frac{1}{6}\ln\left(\frac{1}{18}\right).$$