MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 **MATH 3202** SPRING 2019

SOLUTIONS

[6] 1. (a) All of the traces are ellipses, so this is an ellipsoid.

(b) Two of the traces are hyperbolas and the other is an ellipse, so this is a hyperboloid (of one sheet).

(c) Again, two of the traces and hyperbolas and the other is an ellipse, so this is also a hyperboloid. Furthermore, the constant term is zero so this is specifically a cone.

(d) Two of the traces are parabolas and the third is an ellipse, so this is an elliptic paraboloid.

(e) Two of the traces are parabolas and the third is a hyperbola, so this is a hyperbolic paraboloid.

(f) All of the traces are lines, so this is a plane.

[4] 2. We can treat the surface as a level surface of the function

$$f(x, y, z) = x^2 - xy^2 + z^2$$

where $f(x, y, z) = 13$. Note that when $x = 3$ and $y = -2$, we have

$$9 - 12 + z^2 = 13 \quad \Longrightarrow \quad z^2 = 16$$

and so $z = 4$ since we are given that P is located above the xy-plane (that is, for $z > 0$). Thus P is the point $(3, -2, 4)$. Since

$$\nabla f = \langle 2x - y^2,\ -2xy,\ 2z \rangle$$

a normal to the tangent plane must be

$$\mathbf{n} = \nabla f(3, -2, 4) = \langle 2,\ 12,\ 8 \rangle.$$

The equation of the tangent plane is therefore

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \\ \langle 2,\ 12,\ 8 \rangle \cdot \langle x - 3,\ y + 2,\ z - 4 \rangle &= 0 \\ 2x + 12y + 8z &= 14 \\ x + 6y + 4z &= 7. \end{aligned}$$

[5] 3. A unit vector in the direction of $\mathbf{v}$ is given by

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 1,\ 1\rangle}{\sqrt{2}} = \left\langle \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2} \right\rangle.$$

Furthermore,

$$\nabla f = \langle \sin(2y - x),\ -2\sin(2y - x)\rangle$$

so

$$\nabla f\left(0, \frac{\pi}{12}\right) = \left\langle \frac{1}{2},\ -1 \right\rangle.$$

Thus

$$D_{\mathbf{u}} f(x, y) = \left\langle \frac{1}{2},\ -1 \right\rangle \cdot \left\langle \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2} \right\rangle = -\frac{\sqrt{2}}{4}.$$

[8] 4. The given surface can be written

$$z = 3 - \frac{2}{3}x - 2y = f(x, y)$$

so an increment of surface area is given by

$$\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA = \sqrt{\left(-\frac{2}{3}\right)^2 + (-2)^2 + 1}\, dA = \sqrt{\frac{49}{9}}\, dA = \frac{7}{3}\, dA.$$

In the xy-plane, the surface becomes

$$2x + 6y = 9 \quad \Longrightarrow \quad y = \frac{3}{2} - \frac{1}{3}x.$$

The region of integration is bounded by this diagonal line, as well as the lines $x = 0$ and $y = 0$ (because we are told that S lies in the first octant). Since the diagonal line has $x = \frac{9}{2}$ as its x-intercept, the region of integration is given by

$$0 \leq y \leq \frac{3}{2} - \frac{1}{3}x \quad \text{and} \quad 0 \leq x \leq \frac{9}{2}.$$

Thus

$$
\begin{aligned}
S &= \iint\limits_S \frac{7}{3}\, dA \\
&= \frac{7}{3} \int_0^{\frac{9}{2}} \int_0^{\frac{3}{2}-\frac{1}{3}x} dy\, dx \\
&= \frac{7}{3} \int_0^{\frac{9}{2}} \Big[y \Big]_{y=0}^{y=\frac{3}{2}-\frac{1}{3}x} dx \\
&= \frac{7}{3} \int_0^{\frac{9}{2}} \left(\frac{3}{2} - \frac{1}{3}x \right) dx \\
&= \frac{7}{3} \left[\frac{3}{2}x - \frac{1}{6}x^2 \right]_0^{\frac{9}{2}} \\
&= \frac{7}{3} \left[\frac{27}{4} - \frac{27}{8} \right] \\
&= \frac{63}{8}.
\end{aligned}
$$

Note that we could instead treat the region of integration as being bounded by

$$0 \le x \le \frac{9}{2} - 3y \quad \text{and} \quad 0 \le y \le \frac{3}{2}$$

in which case the iterated integral would be

$$\iint\limits_S \frac{7}{3}\, dA = \frac{7}{3} \int_0^{\frac{3}{2}} \int_0^{\frac{9}{2}-3y} dx\, dy.$$

[5] 5. We have

$$\mathbf{R}_u = \langle 2,\ 0,\ 3v \rangle \quad \text{and} \quad \mathbf{R}_v = \langle 0,\ 2v,\ 3u \rangle.$$

Thus

$$\mathbf{R}_u \times \mathbf{R}_v = \langle -6v^2,\ -6u,\ 4v \rangle$$

and so

$$\|\mathbf{R}_u \times \mathbf{R}_v\| = \sqrt{36v^4 + 36u^2 + 16v^2} = 2\sqrt{9v^4 + 9u^2 + v^2}.$$

Thus an increment of surface area is given by

$$dS = 2\sqrt{9v^4 + 9u^2 + v^2}\, dA.$$

Under the given parametrisation, the integrand becomes

$$\frac{z}{y} = \frac{3uv}{v^2} = \frac{3u}{v}.$$

Hence

$$\iint_S \frac{z}{y}\,dS = \iint_S \frac{3u}{v}\,dS$$
$$= \int_1^4 \int_0^2 \frac{3u}{v} \cdot 2\sqrt{9v^4 + 9u^2 + v^2}\,du\,dv$$
$$= 6\int_1^4 \int_0^2 \frac{u}{v}\sqrt{9v^4 + 9u^2 + v^2}\,du\,dv.$$

Since the limits of integration are all constants, we could equivalently write this as

$$\iint_S \frac{z}{y}\,dS = 6\int_0^2 \int_1^4 \frac{u}{v}\sqrt{9v^4 + 9u^2 + v^2}\,dv\,du.$$

[8] 6. In the xy-plane, the surface $z = x - y$ becomes $0 = x - y$ so $y = x$. Thus the region of integration is bounded by $y = x$ and $y = x^2$, which meet at the points $(0, 0)$ and $(1, 1)$. Since the diagonal line is above the the parabola on the interval $[0, 1]$, we have

$$x^2 \le y \le x \quad \text{and} \quad 0 \le x \le 1.$$

Furthermore,

$$0 \le z \le x - y$$

since the solid E is bounded by the xy-plane (that is, the plane $z = 0$). Thus

$$\iiint_E dV = \int_0^1 \int_{x^2}^x \int_0^{x-y} dz\,dy\,dx$$
$$= \int_0^1 \int_{x^2}^x \Big[z\Big]_{z=0}^{z=x-y} dy\,dx$$
$$= \int_0^1 \int_{x^2}^x (x - y)\,dy\,dx$$
$$= \int_0^1 \left[xy - \frac{1}{2}y^2\right]_{y=x^2}^{y=x} dx$$
$$= \int_0^1 \left[\left(x^2 - \frac{1}{2}x^2\right) - \left(x^3 - \frac{1}{2}x^4\right)\right] dx$$
$$= \int_0^1 \left(\frac{1}{2}x^2 - x^3 + \frac{1}{2}x^4\right) dx$$
$$= \left[\frac{1}{6}x^3 - \frac{1}{4}x^4 + \frac{1}{10}x^5\right]_0^1$$
$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{10}$$
$$= \frac{1}{60}.$$

Note that, since the diagonal line lies to the left of the positive branch of the parabola, we could instead treat the region of integration as being determined by

$$y \le x \le \sqrt{y} \quad \text{and} \quad 0 \le y \le 1.$$

Hence the iterated integral could also be written

$$\iiint\limits_E dV = \int_0^1 \int_y^{\sqrt{y}} \int_0^{x-y} dz\, dx\, dy.$$

[4] 7. Since

$$\nabla z = \langle z_x,\ z_y \rangle$$

we can write

$$\begin{aligned}\nabla(z^n) &= \left\langle \frac{\partial}{\partial x} z^n,\ \frac{\partial}{\partial y} z^n \right\rangle \\ &= \langle n z^{n-1} z_x,\ n z^{n-1} z_y \rangle \\ &= n z^{n-1} \langle z_x,\ z_y \rangle \\ &= n z^{n-1} \nabla z,\end{aligned}$$

as required.