MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 2 **MATH 3202** SPRING 2019

SOLUTIONS

[3] 1. (a) First,

$$\left[\frac{1}{\sqrt{9-t^2}}\right]' = [(9-t^2)^{-\frac{1}{2}}]' = t(9-t^2)^{-\frac{3}{2}} = \frac{t}{(9-t^2)^{\frac{3}{2}}}.$$

Next,

$$\left[\frac{t}{\sqrt{16+t^2}}\right]' = \frac{\sqrt{16+t^2} - t^2(16+t^2)^{-\frac{1}{2}}}{16+t^2} = \frac{16}{(16+t^2)^{\frac{3}{2}}}.$$

Finally,

$$[t\cos(\pi t)]' = \cos(\pi t) - \pi t \sin(\pi t).$$

Hence

$$\mathbf{r}'(t) = \left\langle \frac{t}{(9-t^2)^{\frac{3}{2}}}, \frac{16}{(16+t^2)^{\frac{3}{2}}}, \cos(\pi t) - \pi t \sin(\pi t) \right\rangle.$$

[5] (b) First,

$$\int \frac{1}{\sqrt{9-t^2}}\,dt = \arcsin\left(\frac{t}{3}\right) + C_1.$$

Second, if we let $u = 16 + t^2$ so $\frac{1}{2}\,du = t\,dt$ then

$$\int \frac{t}{\sqrt{16+t^2}}\,dt = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = \sqrt{u} + C_2 = \sqrt{16+t^2} + C_2.$$

Third, using integration by parts,

$$\int t\cos(\pi t)\,dt = \frac{1}{\pi}t\sin(\pi t) - \frac{1}{\pi}\int \sin(\pi t)\,dt = \frac{1}{\pi}t\sin(\pi t) + \frac{1}{\pi^2}\cos(\pi t) + C_3.$$

Hence

$$\int \mathbf{r}(t)\,dt = \left\langle \arcsin\left(\frac{t}{3}\right) + C_1, \sqrt{16+t^2} + C_2, \frac{1}{\pi}t\sin(\pi t) + \frac{1}{\pi^2}\cos(\pi t) + C_3 \right\rangle.$$

[3] (c) Using the results from part (b), we have

$$\int_0^3 \frac{1}{\sqrt{9-t^2}}\,dt = \left[\arcsin\left(\frac{t}{3}\right)\right]_0^3 = \arcsin(1) - \arcsin(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

$$\int_0^3 \frac{t}{\sqrt{16+t^2}}\,dt = \left[\sqrt{16+t^2}\right]_0^3 = 5 - 4 = 1,$$

$$\int_0^3 t\cos(\pi t)\,dt = \left[\frac{1}{\pi}t\sin(\pi t) + \frac{1}{\pi^2}\cos(\pi t)\right]_0^3 = 0 - \frac{1}{\pi^2} - 0 - \frac{1}{\pi^2} = -\frac{2}{\pi^2}.$$

Thus

$$\int_0^3 \mathbf{r}(t)\,dt = \left\langle \frac{\pi}{2}, 1, -\frac{2}{\pi^2} \right\rangle.$$

[3] 2. We have

$$z(t)\mathbf{v}(t) = z(t)\langle f(t), g(t)\rangle = \langle z(t)f(t), z(t)g(t)\rangle$$

so the lefthand side becomes

$$[z(t)\mathbf{v}(t)]' = [\langle z(t)f(t), z(t)g(t)\rangle]' = \langle z'(t)f(t) + z(t)f'(t), z'(t)g(t) + z(t)g'(t)\rangle.$$

On the righthand side, we have

$$z'(t)\mathbf{v}(t) = z'(t)\langle f(t), g(t)\rangle = \langle z'(t)f(t), z'(t)g(t)\rangle$$

and

$$z(t)\mathbf{v}'(t) = z(t)\langle f'(t), g'(t)\rangle = \langle z(t)f'(t), z(t)g'(t)\rangle.$$

Hence

$$z'(t)\mathbf{v}(t) + z(t)\mathbf{v}'(t) = \langle z'(t)f(t) + z(t)f'(t), z'(t)g(t) + z(t)g'(t)\rangle = [z(t)\mathbf{v}(t)]',$$

as required.

[2] 3. (a) We have $\mathbf{r}'(t) = \langle 3t^2 - 5, 2t, -4\rangle$ so $\mathbf{r}'(2) = \langle 7, 4, -4\rangle$ and

$$\|\mathbf{r}'(2)\| = \sqrt{7^2 + 4^2 + (-4)^2} = 9.$$

Hence

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \left\langle \frac{7}{9}, \frac{4}{9}, -\frac{4}{9}\right\rangle.$$

[2] (b) We know that both $\mathbf{r}'(2)$ and $\mathbf{T}'(2)$ are vectors pointing in the direction of the tangent line, and since $\mathbf{r}(2) = \langle -2, 4, -8\rangle$ the point $(-2, 4, -8)$ lies on the tangent line. Thus two possible parametrisations are

$$\mathbf{r}(t) = \langle -2 + 7t, 4 + 4t, -8 - 4t\rangle \quad \text{and} \quad \mathbf{r}(t) = \left\langle -2 + \frac{7}{9}t, 4 + \frac{4}{9}t, -8 - \frac{4}{9}t\right\rangle.$$

[3] 4. (a) We have $\mathbf{r}'(t) = \langle 3t^2 - 3, 2t - 2, 4t^3 - 4t\rangle$ and this is continuous for all t. We need to determine whether there is any value of t for which $\mathbf{r}'(t) = \mathbf{0}$ so first we set

$$3t^2 - 3 = 0 \quad \Longrightarrow \quad t^2 = 1 \quad \Longrightarrow \quad t = \pm 1.$$

For $t = 1$,

$$2t - 2 = 0 \quad \text{and} \quad 4t^3 - 4t = 0.$$

Hence $\mathbf{r}(t)$ <u>is not smooth</u>. (We could also observe that $\mathbf{r}'(-1) \neq \mathbf{0}$, but this does not alter our conclusion.)

[3] (b) We have $\mathbf{r}'(t) = \langle 3t^2 - 3, 2t + 2, 4t^3 + 4t \rangle$ which is also continuous for all t. Again we set

$$3t^2 - 3 = 0 \quad \Longrightarrow \quad t = \pm 1$$

as before. This time, when $t = 1$,

$$2t + 2 = 4 \quad \text{and} \quad 4t^3 + 4t = 8.$$

Furthermore, when $t = -1$,

$$2t + 2 = 0 \quad \text{but} \quad 4t^3 + 4t = -8.$$

Since there is no value of t for which $\mathbf{r}'(t) = \mathbf{0}$, we conclude that $\mathbf{r}(t)$ <u>is smooth</u>.

[4] 5. In order for the two curves to intersect, we must have $t = t^2$, $1 - 2t = -t^2$, and $2t = t^2 + 1$. Thus we solve

$$t = t^2 \quad \Longrightarrow \quad t^2 - t = 0 \quad \Longrightarrow \quad t(t-1) = 0$$

and so $t = 0$ or $t = 1$. If $t = 0$, the other two equations are inconsistent. However, when $t = 1$ we have $1 - 2t = -t^2 = -1$ and $2t = t^2 + 1 = 2$. Hence the curves intersect at $t = 1$.

The angle of intersection θ will be determined by the angle between the tangent lines — or, equivalently, the tangent vectors — to the two curves at $t = 1$. The tangent vectors are

$$\mathbf{r}_1'(t) = \langle 1, -2, 2 \rangle = \mathbf{r}_1'(1) \quad \text{and} \quad \mathbf{r}_2' = \langle 2t, -2t, 2t \rangle \quad \Longrightarrow \quad \mathbf{r}_2'(1) = \langle 2, -2, 2 \rangle.$$

We know that

$$\mathbf{r}_1' \cdot \mathbf{r}_2' = \|\mathbf{r}_1'\| \|\mathbf{r}_2'\| \cos(\theta)$$

and so

$$10 = 3\sqrt{12} \cos(\theta) \quad \Longrightarrow \quad \cos(\theta) = \frac{10}{3\sqrt{12}} = \frac{5\sqrt{3}}{9}.$$

[5] 6. We have

$$\mathbf{r}'(t) = \langle 4t, 3t, 3t^2 \rangle \quad \Longrightarrow \quad \|\mathbf{r}'(t)\| = \sqrt{16t^2 + 9t^2 + 9t^4} = t\sqrt{25 + 9t^2}.$$

Thus, if we let $u = 25 + 9t^2$ so $\frac{1}{18}\,du = t\,dt$,

$$\begin{aligned} L &= \int_0^4 t\sqrt{25 + 9t^2}\,dt \\ &= \frac{1}{18} \int_{25}^{169} \sqrt{u}\,du \\ &= \frac{1}{27} \left[u^{\frac{3}{2}} \right]_{25}^{169} \\ &= \frac{2072}{27}. \end{aligned}$$

[5] 7. (a) We have

$$\mathbf{r}'(t) = \langle e^t\cos(t) - e^t\sin(t), e^t\sin(t) + e^t\cos(t), e^t\rangle = e^t\langle\cos(t) - \sin(t), \sin(t) + \cos(t), 1\rangle$$

so

$$\|\mathbf{r}'(t)\| = e^t\sqrt{[\cos(t) - \sin(t)]^2 + [\sin(t) + \cos(t)]^2 + 1^2} = e^t\sqrt{2\cos^2(t) + 2\sin^2(t) + 1} = e^t\sqrt{3}.$$

Hence

$$\begin{aligned} s(t) &= \int_0^t e^u\sqrt{3}\,du \\ &= \sqrt{3}\Big[e^u\Big]_0^t \\ &= \sqrt{3}(e^t - 1). \end{aligned}$$

[2] (b) From part (a) we have

$$e^t = \frac{\sqrt{3}}{3}s + 1 \quad \Longrightarrow \quad t = \ln\left(\frac{\sqrt{3}}{3}s + 1\right).$$

Thus

$$\mathbf{r}(s) = \left\langle \left(\frac{\sqrt{3}}{3}s + 1\right))\cos\left(\ln\left(\frac{\sqrt{3}}{3}s + 1\right)\right), \left(\frac{\sqrt{3}}{3}s + 1\right))\sin\left(\ln\left(\frac{\sqrt{3}}{3}s + 1\right)\right), \frac{\sqrt{3}}{3}s + 1\right\rangle.$$