MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Test 1	**MATH 3202**	Spring 2019

SOLUTIONS

[2] 1. (a) We have

$$\lim_{t\to\frac{\pi}{4}} 2\cos(t) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

and

$$\lim_{t\to\frac{\pi}{4}} 5\sec^2(t) = \frac{5}{\left(\frac{\sqrt{2}}{2}\right)^2} = 10.$$

Hence

$$\lim_{t\to\frac{\pi}{4}} \mathbf{r}(t) = \left\langle \sqrt{2},\ 10 \right\rangle.$$

[2] (b) We have

$$[2\cos(t)]' = -2\sin(t)$$

and

$$[5\sec^2(t)]' = 10\sec(t)\cdot\sec(t)\tan(t) = 10\sec^2(t)\tan(t).$$

Hence

$$\mathbf{r}'(t) = \langle -2\sin(t),\ 10\sec^2(t)\tan(t)\rangle.$$

[2] (c) We have

$$\int 2\cos(t)\,dt = 2\sin(t) + C_1$$

and

$$\int 5\sec^2(t)\,dt = 5\tan(t) + C_2.$$

Hence

$$\int \mathbf{r}(t)\,dt = \langle 2\sin(t) + C_1,\ 5\tan(t) + C_2\rangle.$$

[3] (d) The parametric equations are

$$x = 2\cos(t) \quad \text{and} \quad y = 5\sec^2(t).$$

Thus $\cos(t) = \frac{1}{2}x$ and hence

$$y = \frac{5}{\cos^2(t)} = \frac{5}{\left(\frac{1}{2}x\right)^2} = \frac{20}{x^2}.$$

However, only part of the curve with this equation is traced out because if $x = 2\cos(t)$ then $-2 \le x \le 2$. Thus $\mathbf{r}(t)$ only traces out of the portion of the curve which lies on this domain.

[6] 2. (a) We have

$$\mathbf{r}'(t) = \left\langle t^{\frac{1}{2}},\ 1,\ \sqrt{2}t^{\frac{1}{2}} \right\rangle$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{t+1+2t} = \sqrt{3t+1}.$$

Then

$$\begin{aligned} s(t) &= \int_0^t \|\mathbf{r}'(u)\|\,du \\ &= \int_0^t \sqrt{3u+1}\,du \\ &= \left[\frac{2}{9}(3u+1)^{\frac{3}{2}}\right]_0^t \\ &= \frac{2}{9}(3t+1)^{\frac{3}{2}} - \frac{2}{9}. \end{aligned}$$

[3] (b) We can rearrange our expression for s to find that

$$t = \frac{1}{3}\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - \frac{1}{3}$$

and therefore

$$\mathbf{r}(s) = \left\langle \frac{2}{3}\left[\frac{1}{3}\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - \frac{1}{3}\right]^{\frac{3}{2}},\ \frac{1}{3}\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - \frac{1}{3},\ \frac{2\sqrt{2}}{3}\left[\frac{1}{3}\left(\frac{9}{2}s+1\right)^{\frac{2}{3}} - \frac{1}{3}\right]^{\frac{3}{2}} \right\rangle.$$

[5] (c) Since $y = t$, we can write

$$\begin{aligned} \int_C \frac{3y+1}{3}\,ds &= \int_0^1 \frac{3t+1}{3}\|\mathbf{r}'(t)\|\,dt \\ &= \int_0^1 \frac{3t+1}{3}\sqrt{3t+1}\,dt \\ &= \frac{1}{3}\int_0^1 (3t+1)^{\frac{3}{2}}\,dt \\ &= \frac{1}{3}\left[\frac{2}{15}(3t+1)^{\frac{5}{2}}\right]_0^1 \\ &= \frac{1}{3}\left[\frac{64}{15} - \frac{2}{15}\right] \\ &= \frac{62}{45}. \end{aligned}$$

[12] 3. (a) We have

$$\mathbf{r}'(t) = \langle 3,\ 4\cos(t),\ -4\sin(t)\rangle$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{9 + 16\cos^2(t) + 16\sin^2(t)} = \sqrt{25} = 5.$$

Thus

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle \frac{3}{5},\ \frac{4}{5}\cos(t),\ -\frac{4}{5}\sin(t)\right\rangle.$$

(b) We have

$$\mathbf{T}'(t) = \left\langle 0,\ -\frac{4}{5}\sin(t),\ -\frac{4}{5}\cos(t)\right\rangle$$

so

$$\|\mathbf{T}'(t)\| = \sqrt{0 + \frac{16}{25}\sin^2(t) + \frac{16}{25}\cos^2(t)} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Thus

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{4}{5}}{5} = \frac{4}{25}.$$

(c) We have

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \langle 0,\ -\sin(t),\ -\cos(t)\rangle.$$

(d) We observe that

$$\mathbf{T}(t)\cdot\mathbf{N}(t) = 0 - \frac{4}{5}\cos(t)\sin(t) + \frac{4}{5}\sin(t)\cos(t) = 0.$$

Since their dot product is zero, the two vectors must be orthogonal.

(e) We have

$$\begin{aligned}\mathbf{B}(t) &= \mathbf{T}(t)\times\mathbf{N}(t)\\ &= \left[-\frac{4}{5}\cos^2(t) - \frac{4}{5}\sin^2(t)\right]\mathbf{i} - \left[-\frac{3}{5}\cos(t) - 0\right]\mathbf{j} + \left[-\frac{3}{5}\sin(t) - 0\right]\mathbf{k}\\ &= -\frac{4}{5}\mathbf{i} + \frac{3}{5}\cos(t)\mathbf{j} - \frac{3}{5}\sin(t)\mathbf{k}\\ &= \left\langle -\frac{4}{5},\ \frac{3}{5}\cos(t),\ -\frac{3}{5}\sin(t)\right\rangle.\end{aligned}$$

[5] 4. If we consider the unit tangent vector $\mathbf{T}$ written in terms of the arclength s, we can use the fact that s is itself a function of t to apply the Chain Rule:

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\cdot\frac{ds}{dt} \quad \text{or} \quad \mathbf{T}'(t) = \frac{d\mathbf{T}}{ds}s'(t).$$

However, we know that $s'(t) = \|\mathbf{r}'(t)\|$ so

$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{ds}\|\mathbf{r}'(t)\| \quad \Longrightarrow \quad \frac{d\mathbf{T}}{ds} = \frac{\mathbf{T}'(t)}{\|\mathbf{r}'(t)\|}.$$

Next, we use the fact that

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \quad \Longrightarrow \quad \mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t)$$

to obtain

$$\frac{d\mathbf{T}}{ds} = \frac{\|\mathbf{T}'(t)\|\mathbf{N}(t)}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}\mathbf{N}(t).$$

Finally, we recall that

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

and thus

$$\frac{d\mathbf{T}}{ds} = \kappa(t)\mathbf{N}(t)$$

which is the first Frenet-Serret formula.