Section 2.3

For a function $z = f(x,y)$ the surface area of the corresponding surface is given by

$$A = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA$$

where D is the domain of the surface.

Now suppose we have a surface S defined by the vector

$$\underline{R}(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle.$$

Let D be the domain of $\underline{R}$. Then as long as S is covered exactly once as (u,v) varies over D then the surface area of S is given by

$$A = \iint_D \|\underline{R}_u \times \underline{R}_v\| dA$$

where $\underline{R}_u = \langle f_u(u,v), g_u(u,v), h_u(u,v) \rangle$ and

$$\underline{R}_v = \langle f_v(u,v), g_v(u,v), h_v(u,v) \rangle.$$

eg Find the surface area of the surface S consisting of the portion of the cone $x^2+y^2 = z^2$ which lies above the disc $x^2+y^2 \leq 36$.

We know that the cone can be parametrised by

$$\underline{R}(u,v) = \langle u\cos(v), u\sin(v), u \rangle$$

We must have $u \geq 0$ and $[u\cos(v)]^2 + [u\sin(v)]^2 \leq 36$

$$u^2\cos^2(v) + u^2\sin^2(v) \leq 36$$

$$u^2 \leq 36 \rightarrow u \leq 6$$

So $0 \leq u \leq 6$.

In order to trace out the cone exactly once, we must have $0 \leq v \leq 2\pi$.

Next, $\underline{R}_u = \langle \cos(v), \sin(v), 1 \rangle$

$\underline{R}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

Then $\underline{R}_u \times \underline{R}_v = \langle u\cos(v), u\sin(v), -u \rangle$

$\|\underline{R}_u \times \underline{R}_v\| = \sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} = \sqrt{2}u$

Finally, $A = \int_0^{2\pi} \int_0^6 \sqrt{2}u \, du \, dv$

$= 18\sqrt{2}\int_0^{2\pi} dv \quad \boxed{= 36\sqrt{2}\pi}$

Now consider a surface S which is the graph of a function $z = g(x,y)$ and let D be the projection of S onto the xy-plane. Then the <u>surface integral</u> of a function $f(x,y,z)$ over the surface S is given by

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{[g_x(x,y)]^2 + [g_y(x,y)]^2 + 1}\,dA$$

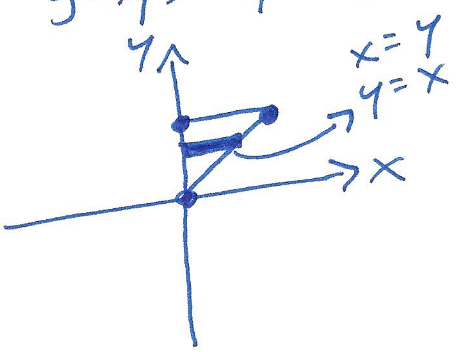
eg Evaluate $\iint_S (z - y^2)\,dS$ where S is the portion of $z = y^2 + 3$ which lies above the triangle in the xy-plane with vertices $(0,0)$, $(0,1)$ and $(1,1)$.

First, $z - y^2 = (y^2+3) - y^2 = 3$.

Next, $\sqrt{[g_x(x,y)]^2 + [g_y(x,y)]^2 + 1}$ for $g(x,y) = y^2 + 3$

$= \sqrt{0^2 + (2y)^2 + 1}$

$= \sqrt{4y^2+1}$



The domain D is bounded by

$0 \leq x \leq y$ and $0 \leq y \leq 1$

So $\displaystyle\iint_S (z - y^2)\,dS = \int_0^1 \int_0^y 3\sqrt{4y^2+1}\,dx\,dy$

$\displaystyle = 3\int_0^1 y\sqrt{4y^2+1}\,dy \;\boxed{= \tfrac{1}{4}(5\sqrt{5}-1)}$

Similarly, if S is defined by $\underline{R}(u,v)$ over the domain D then

$$\iint_S f(x,y,z)\,dS = \iint_D f(\underline{R}(u,v))\,\|\underline{R}_u \times \underline{R}_v\|\,dA.$$

eg Evaluate $\displaystyle\iint_S xz\,dS$ where S is the surface consisting of the portion of the cone $x^2+y^2=z^2$ which lies above the disc $x^2+y^2 \leq 36$.

Since $\underline{R}(u,v) = \langle u\cos(v), u\sin(v), u\rangle$ then

$xz = [u\cos(v)]u = u^2\cos(v)$

and we have already found that $\|\underline{R}_u \times \underline{R}_v\| = \sqrt{2}u$ so

$\displaystyle\iint_S xz\,dS = \int_0^{2\pi}\int_0^6 u^2\cos(v)\sqrt{2}u\,du\,dv$

$\displaystyle = \sqrt{2}\int_0^{2\pi}\int_0^6 u^3\cos(v)\,du\,dv \;\boxed{= 0}$

Section 2.4: Triple Integrals

Given a function $f(x,y,z)$ and a three-dimensional box B defined by $[a,b]\times[c,d]\times[e,f]$ we can define the <u>triple integral</u> of $f(x,y,z)$ over B as

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx.$$

eg Evaluate $\iiint_B x^3y^2z\,dV$ where $B: [0,1]\times[0,2]\times[0,3]$.

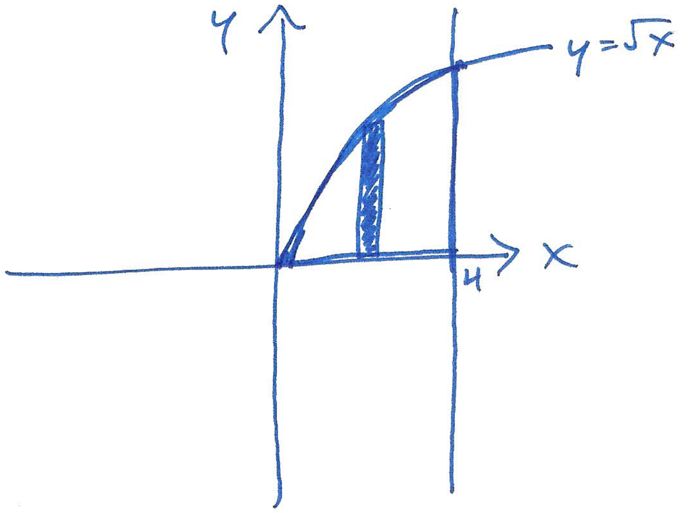
$$\iiint_B x^3y^2z\,dV = \int_0^3\int_0^2\int_0^1 x^3y^2z\,dx\,dy\,dz$$

$$= \int_0^1\int_0^2\int_0^3 x^3y^2z\,dz\,dy\,dx$$

$$= \int_0^2\int_0^3\int_0^1 x^3y^2z\,dy\,dz\,dx \quad \boxed{=3}$$

We can generalize the concept of the triple integral to a solid E in three-dimensional space. In these cases, we typically begin by considering the projection of E in the xy-plane. This will allow us to find bounds on x and y in the manner of a double integral. Finally, to find bounds on z (which may include functions of both x and y) we use our understanding of 3-dimensional geometry.

eg Evaluate $\iiint_E xy\,dV$ where E is the solid that lies below the plane $z=x+y+1$ and above the region in the xy-plane bounded by the curves $y=0$, $y=\sqrt{x}$ and $x=4$.



As a Type ① region, the projection of E in the xy-plane is bounded by

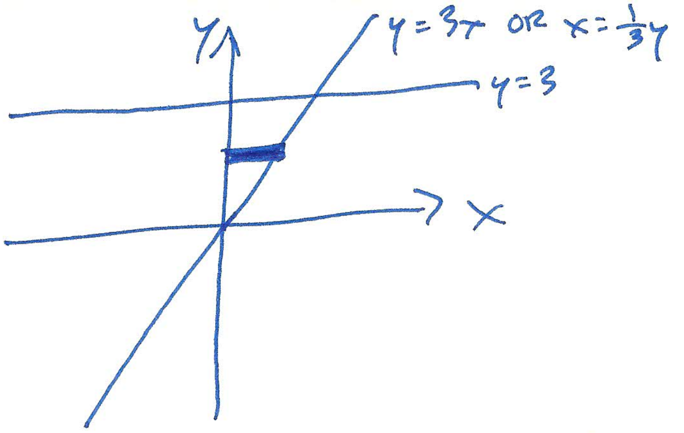
$$0 \le y \le \sqrt{x}$$
$$0 \le x \le 4.$$

Finally, $0 \le z \le x+y+1$

Hence $$\iiint_E xy\,dV = \int_0^4 \int_0^{\sqrt{x}} \int_0^{x+y+1} xy\,dz\,dy\,dx$$
$$= \int_0^4 \int_0^{\sqrt{x}} xy(x+y+1)\,dy\,dx$$
$$= \int_0^4 \left(\tfrac{1}{2}x^3 + \tfrac{1}{3}x^{5/2} + \tfrac{1}{2}x^2\right)dx$$
$$\boxed{= \frac{384}{7}}$$

eg Evaluate $\iiint_E z\,dV$ where E is the solid in the first octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $y=3x$ and $z=0$.

In the xy-plane, $z=0$ so the cylinder has the equation $y^2=9$ so $y=3$. Hence the projection of E in the xy-plane is bounded by $y=3$, $x=0$ and $y=3x$.



The projection is bounded by

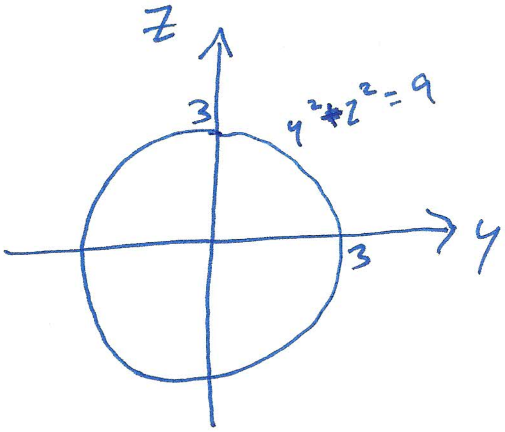
$$0 \le x \le \tfrac{1}{3}y$$
$$0 \le y \le 3$$

Finally, we have

$$0 \le z \le \sqrt{9-y^2}$$

Then $$\iiint_E z\,dV = \int_0^3 \int_0^{\frac{1}{3}y} \int_0^{\sqrt{9-y^2}} z\,dz\,dx\,dy$$
$$= \frac{1}{2}\int_0^3 \int_0^{\frac{1}{3}y} (9-y^2)\,dx\,dy$$
$$= \frac{1}{6}\int_0^3 (9y-y^3)\,dy \quad \boxed{= \frac{27}{8}}$$

As an alternative, we could consider the projection of E onto the yz-plane. Since we are only interested in the first octant, the bounds are



$$0 \le z \le \sqrt{9-y^2} \text{ and } 0 \le y \le 3$$

or $$0 \le y \le \sqrt{9-z^2} \text{ and } 0 \le z \le 3$$

Either way, we have $0 \le x \le \frac{1}{3}y$.

Hence we could instead write
$$\iiint_E z\,dV = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\frac{1}{3}y} z\,dx\,dz\,dy$$
$$= \int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\frac{1}{3}y} z\,dx\,dy\,dz.$$