MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

WORKSHEET | **MATH 2260** | SPRING 2019

SOLUTIONS

1. By the definition of the Laplace transform,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Furthermore,

$$\mathcal{L}\{f(kt)\} = \int_0^\infty e^{-st} f(kt)\, dt.$$

Let $u = kt$ so $du = k\,dt$. Then $t = \frac{1}{k}u$ and $dt = \frac{1}{k}\,du$. When $t = 0$, $u = 0$ and as $t \to \infty$, $u \to \infty$. Thus

$$\mathcal{L}\{f(kt)\} = \frac{1}{k}\int_0^\infty e^{-\frac{su}{k}} f(u)\, du.$$

But the integral is now in the same form as the definition of $F(s)$, except with s replaced by $\frac{s}{k}$. (Remember that the variable of integration here doesn't matter, so there's no problem with the fact that it's now u instead of t.)In other words,

$$\mathcal{L}\{f(kt)\} = \frac{1}{k}F\left(\frac{s}{k}\right),$$

as desired.

2. We can write

$$\begin{aligned}
\mathcal{L}\{\cosh(kt)\} &= \mathcal{L}\left\{\frac{e^{kt} + e^{-kt}}{2}\right\} \\
&= \frac{1}{2}\mathcal{L}\{e^{kt}\} + \frac{1}{2}\mathcal{L}\{e^{-kt}\} \\
&= \frac{1}{2}\left(\frac{1}{s-k}\right) + \frac{1}{2}\left(\frac{1}{s+k}\right) \\
&= \frac{1}{2}\left(\frac{2s}{s^2-k^2}\right) \\
&= \frac{s}{s^2-k^2}.
\end{aligned}$$

3. (a) Observe that

$$F(s) = \frac{1}{2s-6} = \frac{1}{2}\left(\frac{1}{s-3}\right)$$

so

$$f(t) = \frac{1}{2}e^{3t}.$$

(b) We can write

$$F(s) = \frac{4s-1}{s^2+16} = 4\left(\frac{s}{s^2+4^2}\right) - \frac{1}{s^2+4^2} = 4\left(\frac{s}{s^2+4^2}\right) - \frac{1}{4}\left(\frac{4}{s^2+4^2}\right)$$

so

$$f(t) = 4\cos(4t) - \frac{1}{4}\sin(4t).$$

(c) Completing the square gives

$$F(s) = \frac{s-3}{s^2-2s+5} = \frac{s-3}{(s-1)^2+4} = \frac{s-1}{(s-1)^2+2^2} - \frac{2}{(s-1)^2+2^2}$$

so

$$f(t) = e^t\cos(2t) - e^t\sin(2t).$$

(d) We have

$$F(s) = \frac{2}{s} - \frac{1}{s+2} = 2\left(\frac{1}{s}\right) - \frac{1}{s+2}$$

so

$$f(t) = 2(1) - e^{-2t} = 2 - e^{-2t}.$$

4. (a) We take the Laplace transform of both sides of the differential equation to get

$$\begin{aligned} s\mathcal{L}\{y\} - y(0) + 3\mathcal{L}\{y\} &= \frac{2}{s+1} \\ s\mathcal{L}\{y\} - 1 + 3\mathcal{L}\{y\} &= \frac{2}{s+1} \\ (s+3)\mathcal{L}\{y\} &= \frac{2}{s+1} + 1 \\ &= \frac{s+3}{s+1} \\ \mathcal{L}\{y\} &= \frac{1}{s+1} \\ y &= e^{-t}. \end{aligned}$$

(b) We take the Laplace transform of both sides of the differential equation (remembering the use the Shift Theorem for the righthand side) to get

$$\begin{aligned} s\mathcal{L}\{y\} - y(0) - 2\mathcal{L}\{y\} &= \frac{s-2}{(s-2)^2+9} \\ (s-2)\mathcal{L}\{y\} &= \frac{s-2}{(s-2)^2+9} \\ \mathcal{L}\{y\} &= \frac{1}{(s-2)^2+9} \\ &= \frac{1}{3}\left(\frac{3}{(s-2)^2+9}\right) \\ y &= \frac{1}{3}e^{2t}\sin(3t). \end{aligned}$$

(c) We take the Laplace transform of both sides of the differential equation to get

$$\begin{aligned} s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 5[s\mathcal{L}\{y\} - y(0)] + 4\mathcal{L}\{y\} &= 0 \\ s^2\mathcal{L}\{y\} + 2s - 7 - 5s\mathcal{L}\{y\} - 10 + 4\mathcal{L}\{y\} &= 0 \\ (s^2 - 5s + 4)\mathcal{L}\{y\} &= -2s + 17 \\ \mathcal{L}\{y\} &= \frac{-2s + 17}{s^2 - 5s + 4} \\ &= \frac{-2s + 17}{(s-4)(s-1)} \\ &= \frac{3}{s-4} - \frac{5}{s-1} \\ y &= 3e^{4t} - 5e^{t}. \end{aligned}$$

(d) We take the Laplace transform of both sides of the differential equation to get

$$\begin{aligned} s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2[s\mathcal{L}\{y\} - y(0)] + 10\mathcal{L}\{y\} &= 0 \\ s^2\mathcal{L}\{y\} - 3 + 2s\mathcal{L}\{y\} + 10\mathcal{L}\{y\} &= 0 \\ (s^2 + 2s + 10)\mathcal{L}\{y\} &= 3 \\ \mathcal{L}\{y\} &= \frac{3}{s^2 + 2s + 10} \\ &= \frac{3}{(s+1)^2 + 9} \\ y &= e^{-t}\sin(3t). \end{aligned}$$

5. (a) The Laplace transform of the system is

$$\begin{aligned} s\mathcal{L}\{x\} - x(0) &= 3\mathcal{L}\{x\} - 2\mathcal{L}\{y\} \\ s\mathcal{L}\{y\} - y(0) &= 4\mathcal{L}\{x\} + 7\mathcal{L}\{y\}, \end{aligned}$$

which simplifies to

$$\begin{aligned} (s-3)\mathcal{L}\{x\} + 2\mathcal{L}\{y\} &= 3 \\ -4\mathcal{L}\{x\} + (s-7)\mathcal{L}\{y\} &= 2. \end{aligned}$$

The solution of this system is

$$\mathcal{L}\{x\} = \frac{3s - 25}{s^2 - 10s + 29}, \quad \mathcal{L}\{y\} = \frac{2s + 6}{s^2 - 10s + 29}.$$

First, observe that we can write

$$\mathcal{L}\{x\} = \frac{3s - 25}{(s-5)^2 + 4} = \frac{3(s-5)}{(s-5)^2 + 4} - \frac{5(2)}{(s-5)^2 + 4},$$

so

$$x = 3e^{5t}\cos(2t) - 5e^{5t}\sin(2t).$$

Next, we can write

$$\mathcal{L}\{y\} = \frac{2s+6}{(s-5)^2+4} = \frac{2(s-5)}{(s-5)^2+4} + \frac{8(2)}{(s-5)^2+4},$$

so

$$y = 2e^{5t}\cos(2t) + 8e^{5t}\sin(2t).$$

(b) The Laplace transform of the system is

$$\begin{aligned} s\mathcal{L}\{x\} - x(0) &= \mathcal{L}\{x\} - 4\mathcal{L}\{y\} + \frac{1}{s-1} \\ s\mathcal{L}\{y\} - y(0) &= \mathcal{L}\{x\} + \mathcal{L}\{y\} \end{aligned}$$

which simplifies to

$$\begin{aligned} (s-1)\mathcal{L}\{x\} + 4\mathcal{L}\{y\} &= \frac{1}{s-1} - 1 \\ -\mathcal{L}\{x\} + (s-1)\mathcal{L}\{y\} &= 0. \end{aligned}$$

The solution of this system is

$$\mathcal{L}\{x\} = \frac{2-s}{s^2-2s+5} \quad \text{and} \quad \mathcal{L}\{y\} = \frac{2-s}{s^3-3s^2+7s-5}.$$

First, we can write

$$\mathcal{L}\{x\} = \frac{2-s}{(s-1)^2+4} = \frac{-(s-1)}{(s-1)^2+4} + \frac{\frac{1}{2}(2)}{(s-1)^2+4}$$

so

$$x = -e^t\cos(2t) + \frac{1}{2}e^t\sin(2t).$$

Next, we can write

$$\begin{aligned} \mathcal{L}\{y\} &= \frac{2-s}{(s-1)(s^2-2s+5)} = \frac{\frac{1}{4}}{s-1} - \frac{\frac{1}{4}s+\frac{3}{4})}{(s-1)^2+4} \\ &= \frac{\frac{1}{4}}{s-1} - \frac{\frac{1}{4}(s-1)}{(s-1)^2+4} - \frac{\frac{1}{2}(2)}{(s-1)^2+4} \end{aligned}$$

so

$$y = \frac{1}{4}e^t - \frac{1}{4}e^t\cos(2t) - \frac{1}{2}e^t\sin(2t).$$

6. (a) Observe that $e^t = e^{t-2+2} = e^{t-2}e^2$ where e^2 is just a constant. So then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_2(t)e^{t-2}e^2\} = e^2\mathcal{L}\{u_2(t)e^{t-2}\} = e^2(e^{-2s}\mathcal{L}\{e^t\}) = e^{2-2s}\cdot\frac{1}{s-1} = \frac{e^{2-2s}}{s-1}.$$

(b) We can write

$$f(t) = 4 + u_4(t)\cos[3(t-4)]$$

so

$$\mathcal{L}\{f(t)\} = \frac{4}{s} + e^{-4s}\mathcal{L}\{\cos(3t)\} = \frac{4}{s} + \frac{e^{-4s}s}{s^2+9}.$$

(c) We can write

$$f(t) = t[u_1(t) - u_3(t)] = u_1(t)t - u_3(t)t = u_1(t)(t-1) - u_3(t)(t-3) + u_1(t) - 3u_3(t).$$

Thus

$$\mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{t\} - e^{-3s}\mathcal{L}\{t\} + e^{-s}\mathcal{L}\{1\} - 3e^{-3s}\mathcal{L}\{1\} = \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2} + \frac{e^{-s}}{s} - \frac{3e^{-3s}}{s}.$$

7. (a) Since

$$\mathcal{L}\{e^{7t}\} = \frac{1}{s-7},$$

we can write

$$\mathcal{L}\{u_2(t)e^{7(t-2)}\} = \frac{e^{-2s}}{s-7}.$$

Hence $f(t) = u_2(t)e^{7t-14}$.

(b) Completing the square, we have

$$s^2 - 4s + 29 = (s^2 - 4s + 4) + 25 = (s-2)^2 + 25.$$

Then we can write

$$\begin{aligned}\frac{s}{s^2-4s+29} &= \frac{s}{(s-2)^2+25} \\ &= \frac{s-2}{(s-2)^2+25} + \frac{2}{(s-2)^2+25} \\ &= \frac{s-2}{(s-2)^2+25} + \frac{2}{5}\left(\frac{5}{(s-2)^2+25}\right).\end{aligned}$$

By the Shift Theorem, we know that

$$\mathcal{L}\{e^{2t}\cos(5t)\} = \frac{s-2}{(s-2)^2+25} \quad \text{and} \quad \mathcal{L}\{e^{2t}\sin(5t)\} = \frac{5}{(s-2)^2+25}.$$

Hence

$$\mathcal{L}\left\{u_3(t)e^{2(t-3)}\cos[5(t-3)] + \frac{2}{5}u_3(t)e^{2(t-3)}\sin[5(t-3)]\right\} = \frac{s-2}{(s-2)^2+25} + \frac{2}{5}\left(\frac{5}{(s-2)^2+25}\right).$$

So $f(t) = u_3(t)e^{2t-6}\left[\cos(5t-15) + \frac{2}{5}\sin(5t-15)\right]$.

(c) Observe that

$$\frac{1}{s^2-2s+1} = \frac{1}{(s-1)^2}$$

and we know, by the Shift Theorem, that

$$\mathcal{L}\{e^t t\} = \frac{1}{(s-1)^2}.$$

Thus

$$\mathcal{L}\{u_2(t)e^{t-2}(t-2)\} = \frac{e^{-2s}}{(s-1)^2} \quad \text{and} \quad \mathcal{L}\{u_5(t)e^{t-5}(t-5)\} = \frac{e^{-5s}}{(s-1)^2}.$$

We now conclude that $f(t) = u_2(t)e^{t-2}(t-2) - u_5(t)e^{t-5}(t-5)$.

8. The Laplace transform of the given equation is

$$\begin{aligned}
s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 9\mathcal{L}\{y\} &= e^{-\pi s} \cdot \frac{s}{s^2+1} \\
(s^2+9)\mathcal{L}\{y\} &= \frac{e^{-\pi s}s}{s^2+1} + 2 \\
\mathcal{L}\{y\} &= \frac{e^{-\pi s}s}{(s^2+1)(s^2+9)} + \frac{2}{s^2+9} \\
&= \frac{\frac{1}{8}e^{-\pi s}s}{s^2+1} - \frac{\frac{1}{8}e^{-\pi s}s}{s^2+9} + \frac{2}{3}\left(\frac{3}{s^2+9}\right) \\
y &= \frac{1}{8}u_\pi(t)\cos(t-\pi) - \frac{1}{8}u_\pi \cos[3(t-\pi)] + \frac{2}{3}\sin(3t) \\
&= \frac{2}{3}\sin(3t) - \frac{1}{8}u_\pi(t)[\cos(t) - \cos(3t)].
\end{aligned}$$

Here we have simplified our expression for y using the fact that $\cos(x+k\pi) = -\cos(x)$ for any odd integer k.