MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 **MATH 2260** SPRING 2019

SOLUTIONS

[10] 1. (a) The solution of the corresponding homogeneous equation is

$$y_c = C_1 e^{-4t} + C_2 t e^{-4t}.$$

To solve the given equation, we set

$$\begin{aligned} y &= u(t)e^{-4t} + v(t)te^{-4t} \\ \frac{dy}{dt} &= \frac{du}{dt}e^{-4t} - 4u(t)e^{-4t} + \frac{dv}{dt}te^{-4t} + v(t)e^{-4t} - 4v(t)te^{-4t}. \end{aligned}$$

We assume that

$$\frac{du}{dt}e^{-4t} + \frac{dv}{dt}te^{-4t} = 0,$$

so

$$\begin{aligned} \frac{dy}{dt} &= -4u(t)e^{-4t} + v(t)e^{-4t} - 4v(t)te^{-4t} \\ \frac{d^2y}{dt^2} &= -4\frac{du}{dt}e^{-4t} + 16u(t)e^{-4t} + \frac{dv}{dt}e^{-4t} - 8v(t)e^{-4t} - 4\frac{dv}{dt}te^{-4t} + 16v(t)te^{-4t}. \end{aligned}$$

Substituting back into the nonhomogeneous equation, we obtain

$$\begin{aligned} &\left[-4\frac{du}{dt}e^{-4t} + 16u(t)e^{-4t} + \frac{dv}{dt}e^{-4t} - 8v(t)e^{-4t} - 4\frac{dv}{dt}te^{-4t} + 16v(t)te^{-4t}\right] \\ &\qquad + 8\left[-4u(t)e^{-4t} + v(t)e^{-4t} - 4v(t)te^{-4t}\right] + 16[u(t)e^{-4t} + v(t)te^{-4t}] = t^{-2}e^{-4t} \\ &\qquad\qquad -4\frac{du}{dt}e^{-4t} + \frac{dv}{dt}e^{-4t} - 4\frac{dv}{dt}te^{-4t} = t^{-2}e^{-4t}. \end{aligned}$$

Hence we must solve the system of equations

$$\begin{aligned} \frac{du}{dt}e^{-4t} + \frac{dv}{dt}te^{-4t} &= 0 \\ -4\frac{du}{dt}e^{-4t} + \frac{dv}{dt}e^{-4t} - 4\frac{dv}{dt}te^{-4t} &= t^{-2}e^{-4t}. \end{aligned}$$

Multiplying the first equation by 4 and adding the equations together, we see that

$$\begin{aligned} \frac{dv}{dt}e^{-4t} &= t^{-2}e^{-4t} \\ \frac{dv}{dt} &= t^{-2} \\ v(t) &= -t^{-1} + C_2. \end{aligned}$$

Furthermore, we now have

$$\begin{aligned}\frac{du}{dt}e^{-4t} + (t^{-2})te^{-4t} &= 0\\ \frac{du}{dt}e^{-4t} + t^{-1}e^{-4t} &= 0\\ \frac{du}{dt} &= -t^{-1}\\ u(t) &= -\ln(t) + C_1.\end{aligned}$$

Hence the general solution is

$$\begin{aligned}y &= [-\ln(t) + C_1]e^{-4t} + [-t^{-1} + C_2]te^{-4t}\\ &= C_1e^{-4t} + C_2te^{-4t} - \ln(t)e^{-4t} - e^{-4t}.\end{aligned}$$

[10] (b) The solution of the corresponding homogeneous equation is

$$y_c = C_1 \cos\left(\frac{t}{3}\right) + C_2 \sin\left(\frac{t}{3}\right).$$

To solve the given equation, we set

$$\begin{aligned}y &= u(t)\cos\left(\frac{t}{3}\right) + v(t)\sin\left(\frac{t}{3}\right)\\ \frac{dy}{dt} &= \frac{du}{dt}\cos\left(\frac{t}{3}\right) - \frac{1}{3}u(t)\sin\left(\frac{t}{3}\right) + \frac{dv}{dt}\sin\left(\frac{t}{3}\right) + \frac{1}{3}v(t)\cos\left(\frac{t}{3}\right).\end{aligned}$$

We assume that

$$\frac{du}{dt}\cos\left(\frac{t}{3}\right) + \frac{dv}{dt}\sin\left(\frac{t}{3}\right) = 0,$$

so

$$\begin{aligned}\frac{dy}{dt} &= -\frac{1}{3}u(t)\sin\left(\frac{t}{3}\right) + \frac{1}{3}v(t)\cos\left(\frac{t}{3}\right)\\ \frac{d^2y}{dt^2} &= -\frac{1}{3}\frac{du}{dt}\sin\left(\frac{t}{3}\right) - \frac{1}{9}u(t)\cos\left(\frac{t}{3}\right) + \frac{1}{3}\frac{dv}{dt}\cos\left(\frac{t}{3}\right) - \frac{1}{9}v(t)\sin\left(\frac{t}{3}\right).\end{aligned}$$

Substituting back into the nonhomogeneous equation, we obtain

$$\begin{aligned}9\left[-\frac{1}{3}\frac{du}{dt}\sin\left(\frac{t}{3}\right) - \frac{1}{9}u(t)\cos\left(\frac{t}{3}\right) + \frac{1}{3}\frac{dv}{dt}\cos\left(\frac{t}{3}\right) - \frac{1}{9}v(t)\sin\left(\frac{t}{3}\right)\right]&\\ + \left[u(t)\cos\left(\frac{t}{3}\right) + v(t)\sin\left(\frac{t}{3}\right)\right] &= \sec\left(\frac{t}{3}\right)\\ -3\frac{du}{dt}\sin\left(\frac{t}{3}\right) + 3\frac{dv}{dt}\cos\left(\frac{t}{3}\right) &= \sec\left(\frac{t}{3}\right).\end{aligned}$$

Hence we must solve the system of equations

$$\frac{du}{dt}\cos\left(\frac{t}{3}\right)+\frac{dv}{dt}\sin\left(\frac{t}{3}\right)=0$$
$$-3\frac{du}{dt}\sin\left(\frac{t}{3}\right)+3\frac{dv}{dt}\cos\left(\frac{t}{3}\right)=\sec\left(\frac{t}{3}\right).$$

Multiplying the first equation by $3\sin\left(\frac{t}{3}\right)$, the second by $\cos\left(\frac{t}{3}\right)$, and adding the resulting equations, we obtain

$$\begin{aligned}3\frac{dv}{dt}\left[\sin^2\left(\frac{t}{3}\right)+\cos^2\left(\frac{t}{3}\right)\right]&=1\\ 3\frac{dv}{dt}&=1\\ \frac{dv}{dt}&=\frac{1}{3}\\ v(t)&=\frac{t}{3}+C_2.\end{aligned}$$

Furthermore, we now have

$$\begin{aligned}\frac{du}{dt}\cos\left(\frac{t}{3}\right)+\frac{1}{3}\sin\left(\frac{t}{3}\right)&=0\\ \frac{du}{dt}&=-\frac{1}{3}\tan\left(\frac{t}{3}\right)\\ u(t)&=-\ln\left(\sec\left(\frac{t}{3}\right)\right)+C_1.\end{aligned}$$

Hence the general solution is

$$\begin{aligned}y&=\left[-\ln\left(\sec\left(\frac{t}{3}\right)\right)+C_1\right]\cos\left(\frac{t}{3}\right)+\left[\frac{t}{3}+C_2\right]\sin\left(\frac{t}{3}\right)\\ &=C_1\cos\left(\frac{t}{3}\right)+C_2\sin\left(\frac{t}{3}\right)-\ln\left(\sec\left(\frac{t}{3}\right)\right)\cos\left(\frac{t}{3}\right)+\frac{t}{3}\sin\left(\frac{t}{3}\right).\end{aligned}$$

[10] 2. (a) The solution of the corresponding homogeneous equation is

$$y_c=C_1t^4+C_2t^2.$$

To solve the given equation, we set

$$\begin{aligned}y&=u(t)t^4+v(t)t^2\\ \frac{dy}{dt}&=\frac{du}{dt}t^4+4u(t)t^3+\frac{dv}{dt}t^2+2v(t)t.\end{aligned}$$

We assume that

$$\frac{du}{dt}t^4 + \frac{dv}{dt}t^2 = 0,$$

so

$$\begin{aligned}\frac{dy}{dt} &= 4u(t)t^3 + 2v(t)t \\ \frac{d^2y}{dt^2} &= 4\frac{du}{dt}t^3 + 12u(t)t^2 + 2\frac{dv}{dt}t + 2v(t).\end{aligned}$$

Substituting back into the nonhomogeneous equation, we obtain

$$\begin{aligned}t^2\left[4\frac{du}{dt}t^3 + 12u(t)t^2 + 2\frac{dv}{dt}t + 2v(t)\right] - 5t[4u(t)t^3 + 2v(t)t]& \\ + 8[u(t)t^4 + v(t)t^2] &= \sqrt{t} \\ 4\frac{du}{dt}t^5 + 2\frac{dv}{dt}t^3 &= \sqrt{t}.\end{aligned}$$

Hence we must solve the system of equations

$$\begin{aligned}\frac{du}{dt}t^4 + \frac{dv}{dt}t^2 &= 0 \\ 4\frac{du}{dt}t^5 + 2\frac{dv}{dt}t^3 &= \sqrt{t}.\end{aligned}$$

We can solve the first equation for $\frac{du}{dt}$:

$$\frac{du}{dt} = -\frac{dv}{dt}t^{-2}.$$

Substituting into the second equation yields

$$\begin{aligned}4\left(-\frac{dv}{dt}t^{-2}\right)t^5 + 2\frac{dv}{dt}t^3 &= \sqrt{t} \\ -2\frac{dv}{dt}t^3 &= \sqrt{t} \\ \frac{dv}{dt} &= -\frac{1}{2}t^{-\frac{5}{2}} \\ v(t) &= \frac{1}{3}t^{-\frac{3}{2}} + C_2.\end{aligned}$$

Furthermore, we now have

$$\begin{aligned}\frac{du}{dt} &= -\left(-\frac{1}{2}t^{-\frac{5}{2}}\right)t^{-2} \\ &= \frac{1}{2}t^{-\frac{9}{2}} \\ u &= -\frac{1}{7}t^{-\frac{7}{2}} + C_1.\end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= \left[-\frac{1}{7}t^{-\frac{7}{2}} + C_1\right] t^4 + \left[\frac{1}{3}t^{-\frac{3}{2}} + C_2\right] t^2 \\ &= C_1t^4 + C_2t^2 - \frac{1}{7}\sqrt{t} + \frac{1}{3}\sqrt{t} \\ &= C_1t^4 + C_2t^2 + \frac{4}{21}\sqrt{t}. \end{aligned}$$

[10] (b) The solution of the corresponding homogeneous equation is

$$y_c = C_1t^{-1} + C_2t^{-1}\ln(t).$$

To solve the given equation, we set

$$\begin{aligned} y &= u(t)t^{-1} + v(t)t^{-1}\ln(t) \\ \frac{dy}{dt} &= \frac{du}{dt}t^{-1} - u(t)t^{-2} + \frac{dv}{dt}t^{-1}\ln(t) - v(t)t^{-2}\ln(t) + v(t)t^{-2}. \end{aligned}$$

We assume that

$$\frac{du}{dt}t^{-1} + \frac{dv}{dt}t^{-1}\ln(t) = 0,$$

so

$$\begin{aligned} \frac{dy}{dt} &= -u(t)t^{-2} - v(t)t^{-2}\ln(t) + v(t)t^{-2} \\ \frac{d^2y}{dt^2} &= -\frac{du}{dt}t^{-2} + 2u(t)t^{-3} - \frac{dv}{dt}t^{-2}\ln(t) + 2v(t)t^{-3}\ln(t) - 3v(t)t^{-3} + \frac{dv}{dt}t^{-2}. \end{aligned}$$

Substituting back into the nonhomogeneous equation, we obtain

$$\begin{aligned} t^2\left[-\frac{du}{dt}t^{-2} + 2u(t)t^{-3} - \frac{dv}{dt}t^{-2}\ln(t) + 2v(t)t^{-3}\ln(t) - 3v(t)t^{-3} + \frac{dv}{dt}t^{-2}\right] \\ + 3t\left[-u(t)t^{-2} - v(t)t^{-2}\ln(t) + v(t)t^{-2}\right] + \left[u(t)t^{-1} + v(t)t^{-1}\ln(t)\right] &= t\ln(t) \\ -\frac{du}{dt} - \frac{dv}{dt}\ln(t) + \frac{dv}{dt} &= t\ln(t). \end{aligned}$$

Hence we must solve the system of equations

$$\begin{aligned} \frac{du}{dt}t^{-1} + \frac{dv}{dt}t^{-1}\ln(t) &= 0 \\ -\frac{du}{dt} - \frac{dv}{dt}\ln(t) + \frac{dv}{dt} &= t\ln(t). \end{aligned}$$

Multiplying the first equation by t and adding the two equations together, we get

$$\begin{aligned} \frac{dv}{dt} &= t\ln(t) \\ v(t) &= \frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2 + C_2, \end{aligned}$$

where the integration can be carried out by parts. Furthermore, we now have

$$\begin{aligned} \frac{du}{dt}t^{-1} + [t\ln(t)]t^{-1}\ln(t) = 0 \\ \frac{du}{dt} &= -t[\ln(t)]^2 \\ u(t) &= -\frac{1}{2}t^2[\ln(t)]^2 + \frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2 + C_1. \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= \left[-\frac{1}{2}t^2[\ln(t)]^2 + \frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2 + C_1\right]t^{-1} + \left[\frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2 + C_2\right]t^{-1}\ln(t) \\ &= C_1t^{-1} + C_2t^{-1}\ln(t) - \frac{1}{2}t[\ln(t)]^2 + \frac{1}{2}t\ln(t) - \frac{1}{4}t + \frac{1}{2}t[\ln(t)]^2 - \frac{1}{4}t\ln(t) \\ &= C_1t^{-1} + C_2t^{-1}\ln(t) + \frac{1}{4}t\ln(t) - \frac{1}{4}t. \end{aligned}$$

[1] 3. (a) Since $mg = kL$, we have

$$m = \frac{kL}{g} = \frac{5(39.2)}{9.8} = 20.$$

[1] (b) Critical damping will occur when

$$\gamma^2 = 4km = 4(5)(20) = 400 \quad \Longrightarrow \quad \gamma = 20.$$

(Note that we must have $\gamma > 0$.)

[4] (c) We have

$$\begin{aligned} m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku &= g(t) \\ 20\frac{d^2u}{dt^2} + 20\frac{du}{dt} + 5u &= 5e^{-t} \\ 4\frac{d^2u}{dt^2} + 4\frac{du}{dt} + u &= e^{-t}. \end{aligned}$$

The corresponding homogeneous equation is

$$4\frac{d^2u}{dt^2} + 4\frac{du}{dt} + u = 0,$$

with characteristic equation

$$4r^2 + 4r + 1 = 0 \quad \Longrightarrow \quad (2r+1)^2 = 0$$

so $r = -\frac{1}{2}$ (a double root). Thus the complementary solution is

$$u_c = C_1e^{-\frac{1}{2}t} + C_2te^{-\frac{1}{2}t}.$$

Applying the Method of Undetermined Coefficients, we set

$$u_p = Ae^{-t}$$

so

$$\frac{du_p}{dt} = -Ae^{-t} \quad \text{and} \quad \frac{d^2u_p}{dt^2} = Ae^{-t}.$$

Substituting into the non-homogeneous equation yields

$$\begin{aligned} 4[Ae^{-t}] + 4[-Ae^{-t}] + Ae^{-t} &= e^{-t} \\ A &= 1 \end{aligned}$$

and so $u_p = e^{-t}$. Hence the general solution is

$$u = C_1 e^{-\frac{1}{2}t} + C_2 t e^{-\frac{1}{2}t} + e^{-t}.$$

[4] (d) This time we have

$$\begin{aligned} m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku &= g(t) \\ 20\frac{d^2u}{dt^2} + 20\frac{du}{dt} + 5u &= 5e^{-\frac{1}{2}t} \\ 4\frac{d^2u}{dt^2} + 4\frac{du}{dt} + u &= e^{-\frac{1}{2}t}. \end{aligned}$$

Again, the complementary solution is

$$u_c = C_1 e^{-\frac{1}{2}t} + C_2 t e^{-\frac{1}{2}t}.$$

This time, however, $g(t)$ is a solution of the corresponding homogeneous equation so we must treat it as $t^2 e^{-\frac{1}{2}t}$ in order to apply the Method of Undetermined Coefficients. Thus we set

$$u_p = At^2 e^{-\frac{1}{2}t}$$

so

$$\frac{du_p}{dt} = 2Ate^{-\frac{1}{2}t} - \frac{1}{2}At^2 e^{-\frac{1}{2}t}$$

and

$$\frac{d^2u_p}{dt^2} = 2Ae^{-\frac{1}{2}t} - 2Ate^{-\frac{1}{2}t} + \frac{1}{4}At^2 e^{-\frac{1}{2}t}.$$

Substituting into the non-homogeneous equation this time yields

$$\begin{aligned} 4[2Ae^{-\frac{1}{2}t} - 2Ate^{-\frac{1}{2}t} + \frac{1}{4}At^2 e^{-\frac{1}{2}t}] + 4[2Ate^{-\frac{1}{2}t} - \frac{1}{2}At^2 e^{-\frac{1}{2}t}] + At^2 e^{-\frac{1}{2}t} &= e^{-\frac{1}{2}t} \\ 8Ae^{-\frac{1}{2}t} &= e^{-\frac{1}{2}t} \\ A &= \frac{1}{8} \end{aligned}$$

and so $u_p = \frac{1}{8}e^{-t}$. Hence the general solution is

$$u = C_1 e^{-\frac{1}{2}t} + C_2 t e^{-\frac{1}{2}t} + \frac{1}{8} t^2 e^{-\frac{1}{2}t}.$$

Note that we could instead use the Method of Variation of Parameters in parts (c) and (d), but it is far more straightforward to use the Method of Undetermined Coefficients.