MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 6 | **MATH 2260** | Spring 2011

SOLUTIONS

[3] 1. (a) The characteristic equation is

$$r^4 + 4r^3 + 13r^2 + 36r + 36 = (r+2)^2(r^2+9) = 0,$$

which has a double root $r = -2$ and complex conjugate roots $r = \pm 3i$. Hence the general solution is

$$y = C_1 e^{-2t} + C_2 t e^{-2t} + C_3 \cos(3t) + C_4 \sin(3t).$$

[3] (b) The characteristic equation is

$$4r^4 - 13r^3 + 15r^2 - 7r + 1 = (r-1)^3(4r-1),$$

which has a triple root $r = 1$ and a distinct real root $r = \frac{1}{4}$. Hence the general solution is

$$y = C_1 e^t + C_2 t e^t + C_3 t^2 e^t + C_4 e^{\frac{t}{4}}.$$

[4] (c) The characteristic equation is

$$r^3 + r^2 - 6r = 0 = r(r+3)(r-2) = 0,$$

which has three distinct real roots $r = 0$, $r = -3$ and $r = 2$. Hence the general solution is

$$y = C_1 + C_2 e^{-3t} + C_3 e^{2t}.$$

Furthermore, this means

$$\begin{aligned} \frac{dy}{dt} &= -3C_2 e^{-3t} + 2C_3 e^{2t} \\ \frac{d^2y}{dt^2} &= 9C_2 e^{-3t} + 4C_3 e^{2t}. \end{aligned}$$

The initial conditions, then, yield the system of equations

$$\begin{aligned} C_1 + C_2 + C_3 &= 0 \\ -3C_2 + 2C_3 &= 9 \\ 9C_2 + 4C_3 &= 33. \end{aligned}$$

Solving this, we obtain $C_1 = -7$, $C_2 = 1$, $C_3 = 6$ and so the particular solution of the initial value problem is

$$y = -7 + e^{-3t} + 6e^{2t}.$$

[4] 2. (a) The solution of the corresponding homogeneous equation is

$$y_c = C_1 e^{5t} + C_2 e^{3t}.$$

The appropriate form of the particular solution is

$$\begin{aligned} y_p &= A\sin(5t) + B\cos(5t) \\ \frac{dy_p}{dt} &= 5A\cos(5t) - 5B\sin(5t) \\ \frac{d^2y_p}{dt^2} &= -25A\sin(5t) - 25B\cos(5t). \end{aligned}$$

Substituting these into the given equation, we have

$$\begin{aligned} [-25A\sin(5t) - 25B\cos(5t)] - 8[5A\cos(5t) - 5B\sin(5t)] \\ + 15[A\sin(5t) + B\cos(5t)] = 5\sin(5t) \\ (-10A + 40B)\sin(5t) + (-10B - 40A)\cos(5t) = 5\sin(5t). \end{aligned}$$

Equating coefficients, we obtain the system of equations

$$-10A + 40B = 5 \quad \text{and} \quad -10B - 40A = 0,$$

with solution $A = -\dfrac{1}{34}$ and $B = \dfrac{2}{17}$. Hence the general solution of the nonhomogeneous equation is

$$y = y_c + y_p = C_1 e^{5t} + C_2 e^{3t} - \frac{1}{34}\sin(5t) + \frac{2}{17}\cos(5t).$$

[4] (b) The solution of the corresponding homogeneous equation is

$$y_c = C_1 e^{-4t} + C_2 t e^{-4t}.$$

The appropriate form of the particular solution is

$$\begin{aligned} y_p &= At^3 + Bt^2 + Dt + E \\ \frac{dy_p}{dt} &= 3At^2 + 2Bt + D \\ \frac{d^2y_p}{dt^2} &= 6At + 2B. \end{aligned}$$

Substituting these into the given equation, we have

$$\begin{aligned} [6At + 2B] + 8[3At^2 + 2Bt + D] + 16[At^3 + Bt^2 + Dt + E] = 64t^3 \\ 16At^3 + (24A + 16B)t^2 + (6A + 16B + 16D)t + (2B + 8D + 16E) = 64t^3. \end{aligned}$$

Equating coefficients, we obtain the system of equations

$$16A = 64, \quad 24A + 16B = 0, \quad 6A + 16B + 16D = 0, \quad 2B + 8D + 16E = 0,$$

with solution $A = 4$, $B = -6$, $D = \dfrac{9}{2}$ and $E = -\dfrac{3}{2}$. Hence the general solution of the nonhomogeneous equation is

$$y = y_c + y_p = C_1 e^{-4t} + C_2 t e^{-4t} + 4t^3 - 6t^2 + \frac{9}{2}t - \frac{3}{2}.$$

[4] (c) The solution of the corresponding homogeneous equation is

$$y_c = C_1 e^t \cos(2t) + C_2 e^t \sin(2t).$$

The appropriate form of the particular solution is

$$\begin{aligned} y_p &= Ate^{-t} + Be^{-t} \\ \frac{dy_p}{dt} &= -Ate^{-t} + (A-B)e^{-t} \\ \frac{d^2y_p}{dt^2} &= Ate^{-t} + (B-2A)e^{-t}. \end{aligned}$$

Substituting these into the given equation, we have

$$\begin{aligned} [Ate^{-t} + (B-2A)e^{-t}] - 2[-Ate^{-t} + (A-B)e^{-t}] + 5[Ate^{-t} + Be^{-t}] &= 16te^{-t} \\ 8Ate^{-t} + (8B-4A)e^{-t} &= 16te^{-t}. \end{aligned}$$

Equating coefficients, we obtain the system of equations

$$8A = 16 \quad \text{and} \quad 8B - 4A = 0,$$

with solution $A = 2$ and $B = 1$. Hence the general solution of the nonhomogeneous equation is

$$y = y_c + y_p = C_1 e^t \cos(2t) + C_2 e^t \sin(2t) + 2te^{-t} + e^{-t}.$$

[5] (d) The solution of the corresponding homogeneous equation is

$$y_c = C_1 \cos\left(\frac{t}{3}\right) + C_2 \sin\left(\frac{t}{3}\right).$$

Since the nonhomogeneous function $g(t)$ is also a solution of the corresponding homogeneous equation, the appropriate form of the particular solution is

$$\begin{aligned} y_p &= At\sin\left(\frac{t}{3}\right) + Bt\cos\left(\frac{t}{3}\right) \\ \frac{dy_p}{dt} &= \frac{1}{3}At\cos\left(\frac{t}{3}\right) + A\sin\left(\frac{t}{3}\right) - \frac{1}{3}Bt\sin\left(\frac{t}{3}\right) + B\cos\left(\frac{t}{3}\right) \\ \frac{d^2y_p}{dt^2} &= -\frac{1}{9}At\sin\left(\frac{t}{3}\right) + \frac{2}{3}A\cos\left(\frac{t}{3}\right) - \frac{1}{9}Bt\cos\left(\frac{t}{3}\right) - \frac{2}{3}B\sin\left(\frac{t}{3}\right). \end{aligned}$$

Substituting these into the given equation, we have

$$\begin{aligned} 9\left[-\frac{1}{9}At\sin\left(\frac{t}{3}\right) + \frac{2}{3}A\cos\left(\frac{t}{3}\right) - \frac{1}{9}Bt\cos\left(\frac{t}{3}\right) - \frac{2}{3}B\sin\left(\frac{t}{3}\right)\right]& \\ + \left[At\sin\left(\frac{t}{3}\right) + Bt\cos\left(\frac{t}{3}\right)\right] &= \sin\left(\frac{t}{3}\right) \\ 6A\cos\left(\frac{t}{3}\right) - 6B\sin\left(\frac{t}{3}\right) &= \sin\left(\frac{t}{3}\right). \end{aligned}$$

Equating coefficients, we obtain the system of equations

$$6A = 0 \quad \text{and} \quad -6B = 1,$$

with solution $A = 0$ and $B = -\dfrac{1}{6}$. Hence the general solution of the nonhomogeneous equation is

$$y = y_c + y_p = C_1 \cos\left(\frac{t}{3}\right) + C_2 \sin\left(\frac{t}{3}\right) - \frac{1}{6} t \cos\left(\frac{t}{3}\right).$$

[5] (e) The solution of the corresponding homogeneous equation is

$$y_c = C_1 e^{\frac{7t}{2}} + C_2 t e^{\frac{7t}{2}}.$$

Since both $e^{\frac{7t}{2}}$ and $te^{\frac{7t}{2}}$ are the solutions of the corresponding homogeneous equation, the appropriate form of the particular solution is

$$\begin{aligned} y_p &= At^2 e^{\frac{7t}{2}} \\ \frac{dy_p}{dt} &= \frac{7}{2} At^2 e^{\frac{7t}{2}} + 2Ate^{\frac{7t}{2}} \\ \frac{d^2y_p}{dt^2} &= \frac{49}{4} At^2 e^{\frac{7t}{2}} + 14Ate^{\frac{7t}{2}} + 2Ae^{\frac{7t}{2}}. \end{aligned}$$

Substituting these into the given equation, we have

$$4\left[\frac{49}{4} At^2 e^{\frac{7t}{2}} + 14Ate^{\frac{7t}{2}} + 2Ae^{\frac{7t}{2}}\right] - 28\left[\frac{7}{2} At^2 e^{\frac{7t}{2}} + 2Ate^{\frac{7t}{2}}\right] + 49[At^2 e^{\frac{7t}{2}}] = e^{\frac{7t}{2}}$$

$$8Ae^{\frac{7t}{2}} = e^{\frac{7t}{2}}.$$

Equating coefficients, we see that $8A = 1$ so $A = \dfrac{1}{8}$. Hence the general solution of the nonhomogeneous equation is

$$y = y_c + y_p = C_1 e^{\frac{7t}{2}} + C_2 t e^{\frac{7t}{2}} + \frac{1}{8} t^2 e^{\frac{7t}{2}}.$$

[8] 3. The corresponding homogeneous ODE has characteristic equation

$$r^4 + 4r^3 - 5r^2 = r^2(r+5)(r-1) = 0.$$

It has a double root $r = 0$, and distinct real roots $r = -5$ and $r = 1$, so its general solution is

$$y_c = C_1 + C_2 t + C_3 e^{-5t} + C_4 e^t.$$

The appropriate form of the particular solution of the given equation, then, is

$$\begin{aligned} y_p &= Ate^t + Bt^2 \\ \frac{dy_p}{dt} &= Ate^t + Ae^t + 2Bt \\ \frac{d^2y_p}{dt^2} &= Ate^t + 2Ae^t + 2B \\ \frac{d^3y_p}{dt^3} &= Ate^t + 3Ae^t \\ \frac{d^4y_p}{dt^4} &= Ate^t + 4Ae^t. \end{aligned}$$

Substituting these into the nonhomogeneous equation, we obtain

$$\begin{aligned} [Ate^t + 4Ae^t] + 4[Ate^t + 3Ae^t] - 5[Ate^t + 2Ae^t + 2B] &= 18e^t - 10 \\ 6Ae^t - 10B &= 18e^t - 10, \end{aligned}$$

so $6A = 18$ (and hence $A = 3$) and $-10B = -10$ (implying $B = 1$). Thus the solution of the given equation is

$$y = y_c + y_p = C_1 + C_2t + C_3e^{-5t} + C_4e^t + 3te^t + t^2.$$