# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

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ASSIGNMENT 5 | **MATH 2260** | SPRING 2019

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## SOLUTIONS

[2] 1. (a) The characteristic equation is

$$9r^2 + 1 = 0$$

with roots

$$r^2 = -\frac{1}{9}$$
$$r = \pm\frac{1}{3}i.$$

These are complex conjugates with $\lambda = 0$ and $\mu = \frac{1}{3}$. Hence the general solution is

$$y = C_1 \cos\left(\frac{t}{3}\right) + C_2 \sin\left(\frac{t}{3}\right).$$

[2] (b) The characteristic equation is

$$r^2 - 2r + 5 = 0$$

with roots

$$r = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i.$$

These are complex conjugates with $\lambda = 1$ and $\mu = 2$. Hence the general solution is

$$y = C_1 e^t \cos(2t) + C_2 e^t \sin(2t).$$

[2] (c) The characteristic equation is

$$r^2 - 8r + 15 = (r - 5)(r - 3) = 0$$

with roots $r = 5$ and $r = 3$. Since these roots are real and distinct, we know that the general solution is

$$y = C_1 e^{5t} + C_2 e^{3t}.$$

[2] (d) The characteristic equation is

$$r^2 + 8r + 16 = (r + 4)^2 = 0$$

with a double root $r = -4$. Hence the general solution is

$$y = C_1 e^{-4t} + C_2 t e^{-4t}.$$

[2] (e) The characteristic equation is

$$2r^2 + 11r + 12 = (2r+3)(r+4) = 0$$

with roots $r = -\frac{3}{2}$ and $r = -4$. Since these roots are real and distinct, we know that the general solution is

$$y = C_1 e^{-\frac{3}{2}t} + C_2 e^{-4t}.$$

[3] 2. (a) The characteristic equation is

$$r^2 + 10r + 34 = 0$$

with roots

$$r = \frac{-10 \pm \sqrt{100-136}}{2} = -5 \pm 3i.$$

These are complex conjugates with $\lambda = -5$ and $\mu = 3$. Hence the general solution is

$$y = C_1 e^{-5t}\cos(3t) + C_2 e^{-5t}\sin(3t)$$

and so

$$y' = -5C_1 e^{-5t}\cos(3t) - 3C_1 e^{-5t}\sin(3t) - 5C_2 e^{-5t}\sin(3t) + 3C_2 e^{-5t}\cos(3t).$$

From the first initial condition,

$$y(0) = C_1 = 2.$$

From the second initial condition,

$$y'(0) = -5C_1 + 3C_2 = 2 \quad \Longrightarrow \quad 3C_2 = 12 \quad \Longrightarrow \quad C_2 = 4.$$

Hence the particular solution is

$$y = 2e^{-5t}\cos(3t) + 4e^{-5t}\sin(3t).$$

[3] (b) The characteristic equation is

$$4r^2 - 28r + 49 = (2r-7)^2 = 0$$

with a double root $r = \frac{7}{2}$. Hence the general solution is

$$y = C_1 e^{\frac{7}{2}t} + C_2 t e^{\frac{7}{2}t}$$

and so

$$y' = \frac{7}{2}C_1 e^{\frac{7}{2}t} + C_2 e^{\frac{7}{2}t} + \frac{7}{2}C_2 t e^{\frac{7}{2}t}.$$

From the first initial condition,

$$y(0) = C_1 = 0.$$

From the second initial condition,

$$y'(0) = \frac{7}{2}C_1 + C_2 = -4 \quad \Longrightarrow \quad C_2 = -4.$$

Hence the particular solution is

$$y = -4te^{\frac{7}{2}t}.$$

[3] 3. (a) We have

$$\begin{aligned} W(t) &= y_1 \frac{dy_2}{dt} - \frac{dy_1}{dt} y_2 \\ &= \ln(t) \left(\frac{2}{t}\right) - \frac{1}{t}[\ln(t^2)] \\ &= \frac{2\ln(t)}{t} - \frac{\ln(t^2)}{t} \\ &= \frac{\ln(t^2)}{t} - \frac{\ln(t^2)}{t} \\ &= 0. \end{aligned}$$

Since $W(t)$ is zero for all $t$, $y_1$ and $y_2$ do not represent a fundamental set of solutions.

[3] (b) We have

$$\begin{aligned} W(t) &= y_1 \frac{dy_2}{dt} - \frac{dy_1}{dt} y_2 \\ &= \sin(t)[2\cos(2t)] - \cos(t)\sin(2t) \\ &= 2\sin(t)\cos(2t) - \cos(t)\sin(2t) \\ &= 2\sin(t)[2\cos^2(t) - 1] - \cos(t)[2\sin(t)\cos(t)] \\ &= 4\sin(t)\cos^2(t) - 2\sin(t) - 2\sin(t)\cos^2(t) \\ &= 2\sin(t)\cos^2(t) - 2\sin(t) \\ &= 2\sin(t)[\cos^2(t) - 1] \\ &= 2\sin(t)[-\sin^2(t)] \\ &= -2\sin^3(t). \end{aligned}$$

Since $W(t)$ is not identically zero (the function $\sin^3(t)$ is zero only for $t = k\pi$, $k \in \mathbb{Z}$), $y_1$ and $y_2$ represent a fundamental set of solutions.

[6] 4. (a) Assume that $y = v(t)t^2$ so

$$\frac{dy}{dt} = \frac{dv}{dt}t^2 + 2v(t)t \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2v}{dt^2}t^2 + 4\frac{dv}{dt}t + 2v(t).$$

Substituting these into the ODE yields

$$\begin{aligned} 3t^2\left[\frac{d^2v}{dt^2}t^2 + 4\frac{dv}{dt}t + 2v(t)\right] - t\left[\frac{dv}{dt}t^2 + 2v(t)t\right] - 4v(t)t^2 &= 0 \\ 3t^4\frac{d^2v}{dt^2} + 11t^3\frac{dv}{dt} &= 0. \end{aligned}$$

Let $u = \frac{dv}{dt}$ so $\frac{du}{dt} = \frac{d^2v}{dt^2}$. The equation reduces to

$$\begin{aligned} 3t^4\frac{du}{dt} + 11t^3u &= 0 \\ \frac{du}{u} &= -\frac{11}{3}\frac{dt}{t} \\ \int\frac{du}{u} &= -\frac{11}{3}\int\frac{dt}{t} \\ \ln(u) &= -\frac{11}{3}\ln(t) + C_2 \\ u &= C_2t^{-\frac{11}{3}} \\ \frac{dv}{dt} &= C_2t^{-\frac{11}{3}} \\ v &= C_2t^{-\frac{8}{3}} + C_1. \end{aligned}$$

Hence

$$y = (C_2t^{-\frac{8}{3}} + C_1)t^2 = C_1t^2 + C_2t^{-\frac{2}{3}}$$

is the general solution, and

$$y_2 = t^{-\frac{2}{3}}.$$

To verify that $y_1$ and $y_2$ form a fundamental set of solutions, observe that

$$\begin{aligned} W(t) &= y_1\frac{dy_2}{dt} - y_2\frac{dy_1}{dt} \\ &= t^2\left(-\frac{2}{3}t^{-\frac{5}{3}}\right) - t^{-\frac{2}{3}}(2t) \\ &= -\frac{2}{3}t^{\frac{1}{3}} - 2t^{\frac{1}{3}} \\ &= \frac{8}{3}\sqrt[3]{t}, \end{aligned}$$

which is not identically zero.

[6] (b) Assume that $y = v(t)t^{-1}$ so

$$\frac{dy}{dt} = \frac{dv}{dt}t^{-1} - v(t)t^{-2} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2v}{dt^2}t^{-1} - 2\frac{dv}{dt}t^{-2} + 2v(t)t^{-3}.$$

Substituting these into the ODE yields

$$t^2\left[\frac{d^2v}{dt^2}t^{-1} - 2\frac{dv}{dt}t^{-2} + 2v(t)t^{-3}\right] + 3t\left[\frac{dv}{dt}t^{-1} - v(t)t^{-2}\right] + v(t)t^{-1} = 0$$

$$t\frac{d^2v}{dt^2} + \frac{dv}{dt} = 0.$$

Let $u = \frac{dv}{dt}$ so $\frac{du}{dt} = \frac{d^2v}{dt^2}$. The equation reduces to

$$\begin{aligned} t\frac{du}{dt} + u &= 0 \\ \int \frac{du}{u} &= -\int \frac{dt}{t} \\ \ln(u) &= -\ln(t) + C_2 \\ &= \ln(t^{-1}) + C_2 \\ u &= \frac{C_2}{t} \\ \frac{dv}{dt} &= \frac{C_2}{t} \\ v &= C_2 \ln(t) + C_1. \end{aligned}$$

Hence

$$y = [C_2 \ln(t) + C_1] \cdot \frac{1}{t} = \frac{C_1}{t} + \frac{C_2 \ln(t)}{t}$$

is the general solution, and

$$y_2 = \frac{\ln(t)}{t}.$$

To verify that $y_1$ and $y_2$ form a fundamental set of solutions, observe that

$$\begin{aligned} W(t) &= y_1 \frac{dy_2}{dt} - y_2 \frac{dy_1}{dt} \\ &= \frac{1}{t}\left(\frac{1}{t^2} - \frac{\ln(t)}{t^2}\right) - \frac{\ln(t)}{t}\left(-\frac{1}{t^2}\right) \\ &= \frac{1}{t^3}, \end{aligned}$$

which is never zero.

[6] (c) Assume that $y = v(t)\sqrt{t}$ so

$$\frac{dy}{dt} = \frac{dv}{dt}\sqrt{t} + \frac{1}{2}v(t)t^{-\frac{1}{2}} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2v}{dt^2}\sqrt{t} + \frac{dv}{dt}t^{-\frac{1}{2}} - \frac{1}{4}v(t)t^{-\frac{3}{2}}.$$

Substituting these into the ODE yields

$$\begin{aligned} 2t^2\left[\frac{d^2v}{dt^2}\sqrt{t} + \frac{dv}{dt}t^{-\frac{1}{2}} - \frac{1}{4}v(t)t^{-\frac{3}{2}}\right] + 5t\left[\frac{dv}{dt}\sqrt{t} + \frac{1}{2}v(t)t^{-\frac{1}{2}}\right] - 2v(t)\sqrt{t} &= 0 \\ 2t^{\frac{5}{2}}\frac{d^2v}{dt^2} + 7t^{\frac{3}{2}}\frac{dv}{dt} &= 0. \end{aligned}$$

Let $u = \frac{dv}{dt}$ so $\frac{du}{dt} = \frac{d^2v}{dt^2}$. The equation reduces to

$$\begin{aligned}
2t^{\frac{5}{2}}\frac{du}{dt} + 7t^{\frac{3}{2}}u = 0 \\
\int \frac{du}{u} &= -\frac{7}{2}\int \frac{dt}{t} \\
\ln(u) &= -\frac{7}{2}\ln(t) + C_2 \\
&= \ln(t^{-\frac{7}{2}}) + C_2 \\
u &= C_2 t^{-\frac{7}{2}} \\
v &= C_2 t^{-\frac{5}{2}} + C_1.
\end{aligned}$$

Hence

$$y = (C_2 t^{-\frac{5}{2}} + C_1)\sqrt{t} = C_1\sqrt{t} + \frac{C_2}{t^2}$$

is the general solution, and

$$y_2 = \frac{1}{t^2}.$$

To verify that $y_1$ and $y_2$ form a fundamental set of solutions, observe that

$$\begin{aligned}
W(t) &= y_1\frac{dy_2}{dt} - y_2\frac{dy_1}{dt} \\
&= \sqrt{t}\left(-\frac{2}{t^3}\right) - \frac{1}{t^2}\left(\frac{1}{2}t^{-\frac{1}{2}}\right) \\
&= -\frac{5}{2}t^{-\frac{5}{2}},
\end{aligned}$$

which is never zero.