MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 4	**MATH 2260**	Spring 2019

SOLUTIONS

[3] 1. (a) We can rewrite the equation as

$$\frac{dy}{dt} - \frac{t+1}{t^3}y = \frac{1}{t^3(t-4)}$$

so

$$p(t) = -\frac{t+1}{t^3} \quad \text{and} \quad g(t) = \frac{1}{t^3(t-4)}.$$

Both functions are discontinuous at $t = 0$, while $g(t)$ is also discontinuous at $t = 4$. Hence the possible intervals of definition are $t < 0$, $0 < t < 4$, and $t > 4$. The initial condition occurs at $t = 2$, however, so the interval of definition for the solution of this IVP is $0 < t < 4$.

[3] (b) The ODE is the same as the one given in part (a), so it has the same possible intervals of definition. In this case, the initial condition occurs at $t = -3$, however, so the required interval of definition is $t < 0$.

[3] (c) Here,

$$p(t) = \tan(t) \quad \text{and} \quad g(t) \equiv 0.$$

Note that $p(t)$ is discontinuous for all $t = \frac{k\pi}{2}$ where k is an odd integer, while $g(t)$ is continuous everywhere. Since the initial condition occurs at $t = \pi$, the interval of definition is

$$\frac{\pi}{2} < t < \frac{3\pi}{2}.$$

[3] (d) We rewrite the equation as

$$\frac{dy}{dt} - \frac{1}{(3-t)(3+t)}y = \frac{\cos(3t)}{(3-t)(3+t)}.$$

Observe that both $p(t)$ and $g(t)$ are discontinuous at $t = -3$ and $t = 3$, so the possible intervals of definition are $t < -3$, $-3 < t < 3$, and $t > 3$. Since the initial condition occurs at $t = 0$, the required interval of definition in this instance is $-3 < t < 3$.

[3] 2. (a) We can rewrite the equation as

$$\frac{dy}{dt} = \frac{\sin(t) - y^2 - 1}{ty - 2y + 4t - 8} = \frac{\sin(t) - y^2 - 1}{(t-2)(y+4)} = f(t, y),$$

so

$$\frac{\partial f}{\partial y} = \frac{1 - \sin(t) - y^2 - 8y}{(t-2)(y+4)^2}.$$

Note that both of these are discontinuous at $t = 2$ and $y = -4$. Hence a rectangle around the point $(0, 3)$ which avoids these values is given by

$$-1 < t < 1, \quad 0 < y < 6.$$

[3] (b) A rectangle around the point $(5,-5)$ which avoids $t=2$ and $y=-4$ is given by

$$4 < t < 6, \quad -10 < y < -\frac{9}{2}.$$

[3] (c) Because the initial condition occurs at $y=-4$, which is a discontinuity of $f(t,y)$, we cannot draw a rectangle around $(1,-4)$ in which $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous. Hence the requirements of the theorem cannot be satisfied.

[2] 3. (a) The only difference between this model and the original model we derived in class is that the rate of inflow/outflow is now rt rather than r. Thus the appropriate differential equation is

$$\frac{dQ}{dt} = krt - \frac{rtQ(t)}{V}.$$

[5] (b) It helps to first rewrite this equation as

$$\frac{dQ}{dt} + \frac{rt}{V}Q(t) = krt.$$

This is still a linear equation with

$$p(t) = \frac{rt}{V} \quad \text{and} \quad g(t) = krt.$$

Thus an appropriate integrating factor is

$$\mu = e^{\int \frac{rt}{V}\,dt} = e^{\frac{rt^2}{2V}}.$$

Hence the equation becomes

$$\begin{aligned} \frac{d}{dt}\left[e^{\frac{rt^2}{2V}}Q(t)\right] &= krte^{\frac{rt^2}{2V}} \\ e^{\frac{rt^2}{2V}}Q(t) &= kr\int te^{\frac{rt^2}{2V}}\,dt \\ &= kVe^{\frac{rt^2}{2V}} + C, \end{aligned}$$

where the integral can be evaluated by u-substitution. Thus the general solution is

$$Q(t) = kV + Ce^{-\frac{rt^2}{2V}}.$$

Since $Q(0) = Q_0$, we have

$$\begin{aligned} Q_0 &= kV + C \\ C &= Q_0 - kV \end{aligned}$$

and so the particular solution is

$$Q(t) = kV + (Q_0 - kV)e^{-\frac{rt^2}{2V}}.$$

As $t \to \infty$ we can observe that $Q(t) \to kV$, and so the salt concentration approaches $\frac{kV}{V} = k$. This is the same result we obtained for our original model. Thus the change to the rate of inflow/outflow has <u>no effect</u> on the long-term behaviour of the salt concentration.

[4] 4. (a) We set

$$\begin{aligned}\frac{dy}{dt} &= 0\\ ry\left(1-\frac{y}{K}\right)-Fy &= 0\\ (r-F)y-\frac{r}{K}y^2 &= 0\\ y\left[(r-F)-\frac{r}{K}y\right] &= 0,\end{aligned}$$

so either $y = 0$ or $y = K\left(1-\frac{F}{r}\right)$. These are the fixed points of Equation (3). Observe that the number of fixed points is the same as Equation (1), and indeed 0 is a fixed point in both cases. Now, however, the fixed point $y = K$ of Equation (1) has been replaced by the fixed point $y = K\left(1-\frac{F}{r}\right)$; note that this means that the fixed point has been made smaller (closer to 0) since

$$1-\frac{F}{R} < 1.$$

[5] (b) See Figure 1. We can see that $y = 0$ is an unstable fixed point, while $y = K\left(1-\frac{F}{r}\right)$ is an asymptotically stable fixed point. Thus the stability of each fixed point of Equation (3) is the same as the stability of the corresponding fixed points of Equation (1).

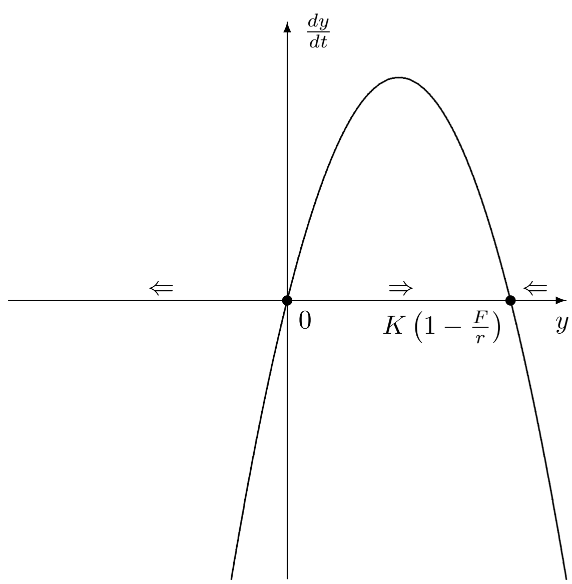

Figure 1: The graph of $\frac{dy}{dt}$ vs. y for Equation (3).

[3] (c) The effect of increasing F is to shift the positive fixed point closer and closer to 0. The codfish population will be viable as long as this fixed point remains positive, so the value

F^* will be the value of F that makes the second fixed point equal to 0. We set

$$\begin{aligned} K\left(1 - \frac{F^*}{r}\right) &= 0 \\ 1 - \frac{F^*}{r} &= 0 \\ F^* &= r. \end{aligned}$$

Hence the population of codfish will be viable as long as $F^* < r$.