MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 3 **MATH 2260** SPRING 2019

SOLUTIONS

[3] 1. (a) We can rewrite the given equation as

$$\begin{aligned}\frac{1}{(y+2)(y-3)}\,dy &= \tan(t)\,dt\\ \int \frac{1}{(y+2)(y-3)}\,dy &= \int \tan(t)\,dt\\ \int\left(\frac{\frac{1}{5}}{y-3}-\frac{\frac{1}{5}}{y+2}\right)dy &= -\ln(\cos(t))+C\\ \frac{1}{5}\ln(y-3)-\frac{1}{5}\ln(y+2) &= -\ln(\cos(t))+C.\end{aligned}$$

This is sufficient, but note that we can simplify this to an extent. If we replace C with $\ln(C)$ and use the properties of logarithms, we have

$$\begin{aligned}\frac{1}{5}\left[\ln(y-3)-\ln(y+2)\right] &= -\ln(\cos(t))+\ln(C)\\ \ln\left(\frac{y-3}{y+2}\right)^{\frac{1}{5}} &= \ln\left(\frac{C}{\cos(t)}\right)\\ \left(\frac{y-3}{y+2}\right)^{\frac{1}{5}} &= \frac{C}{\cos(t)},\end{aligned}$$

which is a bit simpler than our original answer.

[3] (b) The equation becomes

$$\begin{aligned}\frac{t}{1+y^2}\frac{dy}{dt} &= \sqrt{t}-\frac{t^2}{e^t}\\ \frac{1}{1+y^2}\,dy &= \left(t^{-\frac{1}{2}}-te^{-t}\right)dt\\ \int\frac{1}{1+y^2}\,dy &= \int\left(t^{-\frac{1}{2}}-te^{-t}\right)dt\\ \arctan(y) &= 2\sqrt{t}+te^{-t}+e^{-t}+C,\end{aligned}$$

where the integral on the righthand side can be evaluated using integration by parts with $u = t$ and $dv = e^{-t}\,dt$.

[2] 2. (a) We can rewrite the equation as

$$\frac{dy}{dt} = \frac{y+t}{y-t}.$$

So let

$$\begin{aligned} f(t,y) &= \frac{y+t}{y-t} \\ f(kt,ky) &= \frac{ky+kt}{ky-kt} \\ &= \frac{k(y+t)}{k(y-t)} \\ &= \frac{y+t}{y-t} \\ &= f(t,y). \end{aligned}$$

Hence this equation is homogeneous.

[4] (b) Let $y = vt$ so $\frac{dy}{dt} = v + t\frac{dv}{dt}$. The equation becomes

$$\begin{aligned} v + t\frac{dv}{dt} &= \frac{vt+t}{vt-t} \\ &= \frac{v+1}{v-1} \\ t\frac{dv}{dt} &= \frac{v+1}{v-1} - v \\ &= \frac{-v^2+2v+1}{v-1} \\ \frac{v-1}{-v^2+2v+1}\,dv &= \frac{1}{t}\,dt \\ \int \frac{v-1}{-v^2+2v+1}\,dv &= \int \frac{1}{t}\,dt \\ -\frac{1}{2}\ln(-v^2+2v+1) &= \ln(t) + C \\ (-v^2+2v+1)^{-\frac{1}{2}} &= Ct \\ -v^2+2v+1 &= \frac{C}{t^2} \\ -\frac{y^2}{t^2} + \frac{2y}{t} + 1 &= \frac{C}{t^2} \\ -y^2 + 2ty + t^2 &= C. \end{aligned}$$

Note that the integration on the lefthand side can be performed by u-substitution, with $u = -v^2 + 2v + 1$.

[5] 3. We rewrite the equation as

$$y^{-\frac{1}{2}}\frac{dy}{dt} + 2\cos(t)y^{\frac{1}{2}} = 2\cos(t).$$

This is a Bernoulli equation with $n = \frac{1}{2}$, so we set

$$v = y^{\frac{1}{2}} \quad \text{and} \quad 2\frac{dv}{dt} = y^{-\frac{1}{2}}\frac{dy}{dt}.$$

The equation becomes

$$\begin{aligned} 2\frac{dv}{dt} + 2\cos(t)v &= 2\cos(t) \\ \frac{dv}{dt} + \cos(t)v &= \cos(t). \end{aligned}$$

This is a linear ODE with integrating factor

$$\mu = e^{\int \cos(t)\,dt} = e^{\sin(t)}.$$

Multiplying both sides of the equation by μ, we obtain

$$\begin{aligned} \frac{d}{dt}[e^{\sin(t)}v] &= e^{\sin(t)}\cos(t) \\ e^{\sin(t)}v &= \int e^{\sin(t)}\cos(t) \\ e^{\sin(t)}v &= e^{\sin(t)} + C \\ v &= 1 + Ce^{-\sin(t)} \\ y^{\frac{1}{2}} &= 1 + Ce^{-\sin(t)} \\ y &= [1 + Ce^{-\sin(t)}]^2. \end{aligned}$$

[4] 4. (a) Here,

$$M(t,y) = \frac{y^4\csc(\sqrt{t})\cot(\sqrt{t})}{\sqrt{t}} - e^t \quad \text{and} \quad N(t,y) = -8y^3\csc(\sqrt{t})$$

so

$$\frac{\partial M}{\partial y} = \frac{4y^3\csc(\sqrt{t})\cot(\sqrt{t})}{\sqrt{t}} \quad \text{and} \quad \frac{\partial N}{\partial t} = \frac{4y^3\csc(\sqrt{t})\cot(\sqrt{t})}{\sqrt{t}}.$$

Hence this equation is exact.

Now we know that

$$\begin{aligned}\psi(t,y) &= -8\int y^3\csc(\sqrt{t})\,dy\\ &= -2y^4\csc(\sqrt{t}) + f(t)\\ \frac{\partial\psi}{\partial t} &= \frac{y^4\csc(\sqrt{t})\cot(\sqrt{t})}{\sqrt{t}} + f'(t)\\ \frac{y^4\csc(\sqrt{t})\cot(\sqrt{t})}{\sqrt{t}} - e^t &= \frac{y^4\csc(\sqrt{t})\cot(\sqrt{t})}{\sqrt{t}} + f'(t)\\ -e^t &= f'(t)\\ -e^t + C &= f(t).\end{aligned}$$

Hence the general solution is

$$-2y^4\csc(\sqrt{t}) - e^t = C.$$

[5] (b) Here,

$$M(t,y) = ty^2 - y \quad \text{and} \quad N(t,y) = t + 2y^3$$

so

$$\frac{\partial M}{\partial y} = 2ty - 1 \quad \text{and} \quad \frac{\partial N}{\partial t} = 1.$$

Hence this equation is not exact. However, note that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{-M} = -\frac{2ty-2}{ty^2-y} = -\frac{2(ty-1)}{y(ty-1)} = -\frac{2}{y},$$

which is purely a function of y. Thus an appropriate integrating factor is

$$\mu = e^{-2\int \frac{dy}{y}} = e^{-2\ln(y)} = y^{-2}.$$

Multiplying both sides of the equation by μ, we obtain

$$t - y^{-1} - (ty^{-2} + 2y)\frac{dy}{dt} = 0.$$

Now

$$M^*(t,y) = t - y^{-1} \quad \text{and} \quad N^*(t,y) = ty^{-2} + 2y,$$

so

$$\frac{\partial M^*}{\partial y} = y^{-2} \quad \text{and} \quad \frac{\partial N^*}{\partial t} = y^{-2},$$

so the equation has indeed been made exact. Then

$$\begin{aligned}\psi(t,y) &= \int (t - y^{-1})\,dt \\ &= \frac{1}{2}t^2 - ty^{-1} + f(y) \\ \frac{\partial\psi}{\partial y} &= ty^{-2} + f'(y) \\ ty^{-2} + 2y &= ty^{-2} + f'(y) \\ 2y &= f'(y) \\ y^2 + C &= f(y).\end{aligned}$$

Thus the general solution is

$$\frac{1}{2}t^2 - ty^{-1} + y^2 = C.$$

[5] (c) Here,

$$M(t,y) = e^y \sin(t) \quad \text{and} \quad N(t,y) = e^y$$

so

$$\frac{\partial M}{\partial y} = e^y \sin(t) \quad \text{and} \quad \frac{\partial N}{\partial t} = 0.$$

Hence this equation is not exact. But observe that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N} = \frac{e^y \sin(t)}{e^y} = \sin(t),$$

which is purely a function of t. Thus a suitable integrating factor is

$$\mu = e^{\int \sin(t)\,dt} = e^{-\cos(t)}.$$

Multiplying both sides of the ODE by μ, we get

$$e^y e^{-\cos(t)} \sin(t) + e^y e^{-\cos(t)} \frac{dy}{dt} = 0.$$

Now

$$M^*(t,y) = e^y e^{-\cos(t)} \sin(t) \quad \text{and} \quad N^*(t,y) = e^y e^{-\cos(t)}$$

so

$$\frac{\partial M^*}{\partial y} = e^y e^{-\cos(t)} \sin(t) \quad \text{and} \quad \frac{\partial N^*}{\partial t} = e^y e^{-\cos(t)} \sin(t),$$

so the equation is now exact. We have

$$\begin{aligned}\psi(t,y) &= \int e^y e^{-\cos(t)}\,dy \\ &= e^y e^{-\cos(t)} + f(t) \\ \frac{\partial\psi}{\partial t} &= e^y e^{-\cos(t)}\sin(t) + f'(t) \\ e^y e^{-\cos(t)}\sin(t) &= e^y e^{-\cos(t)}\sin(t) + f'(t) \\ 0 &= f'(t) \\ C &= f(t).\end{aligned}$$

Thus the general solution is

$$e^y e^{-\cos(t)} = C.$$

Alternatively, we could use the fact that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{-M} = \frac{e^y\sin(t)}{-e^y\sin(t)} = -1,$$

which can be viewed as purely a function of y. Then the integrating factor is

$$\mu = e^{\int(-1)\,dy} = e^{-y}.$$

Multiplying both sides of the ODE by μ yields

$$\sin(t) + \frac{dy}{dt} = 0.$$

In this case,

$$M^*(t,y) = \sin(t) \quad \text{and} \quad N^*(t,y) = 1$$

so

$$\frac{\partial M^*}{\partial y} = 0 \quad \text{and} \quad \frac{\partial N^*}{\partial t} = 0.$$

Again, the equation has been made exact. So

$$\begin{aligned}\psi(t,y) &= \int dy \\ &= y + f(t) \\ \frac{\partial\psi}{\partial t} &= f'(t) \\ \sin(t) &= f'(t) \\ -\cos(t) + C &= f(t).\end{aligned}$$

Hence the general solution is

$$y - \cos(t) = C.$$

(Note that these two solutions are completely equivalent: just exponentiate the second version and replace e^C by C to obtain the first version.)

[4] 5. (a) This equation is nonlinear and is neither separable nor a Bernoulli equation, so we should determine if it's homogeneous. The equation can be rewritten

$$\frac{dy}{dt} = \sec\left(\frac{y}{t}\right) + \frac{y}{t}$$

so

$$\begin{aligned} f(t,y) &= \sec\left(\frac{y}{t}\right) + \frac{y}{t} \\ f(kt,ky) &= \sec\left(\frac{ky}{kt}\right) + \frac{ky}{kt} \\ &= \sec\left(\frac{y}{t}\right) + \frac{y}{t} \\ &= f(t,y). \end{aligned}$$

Let $y = vt$ so $\dfrac{dy}{dt} = v + t\dfrac{dv}{dt}$. The equation becomes

$$\begin{aligned} v + t\frac{dv}{dt} &= \sec(v) + v \\ t\frac{dv}{dt} &= \sec(v) \\ \cos(v)\,dv &= \frac{1}{t}\,dt \\ \int \cos(v)\,dv &= \int \frac{1}{t}\,dt \\ \sin(v) &= \ln(t) + C \\ \sin\left(\frac{y}{t}\right) &= \ln(t) + C. \end{aligned}$$

To find the particular solution, we substitute the initial condition into the general solution:

$$\begin{aligned} \sin\left(\frac{\pi}{6}\right) &= \ln(1) + C \\ \frac{1}{2} &= C. \end{aligned}$$

Hence the particular solution is

$$\sin\left(\frac{y}{t}\right) = \ln(t) + \frac{1}{2}.$$

[5] (b) This is a nonlinear equation which is not separable, and it's easy to see that it is not homogeneous or a Bernoulli equation. Thus we must hope that it is either exact, or can be made exact. Here,

$$M(t,y) = ye^{ty} + 1 \quad \text{and} \quad N(t,y) = -1 + te^{ty}$$

so

$$\frac{\partial M}{\partial y} = e^{ty} + tye^{ty} \quad \text{and} \quad \frac{\partial N}{\partial t} = e^{ty} + tye^{ty}.$$

Hence this equation is exact.
Now we know that

$$\begin{aligned} \psi(t,y) &= \int [-1 + te^{ty}]\, dy \\ &= -y + e^{ty} + f(t) \\ \frac{\partial \psi}{\partial t} &= ye^{ty} + f'(t) \\ ye^{ty} + 1 &= ye^{ty} + f'(t) \\ 1 &= f'(t) \\ t + C &= f(t). \end{aligned}$$

Hence the general solution is

$$-y + e^{ty} + t = C.$$

Finally, we substitute $t = 0$ and $y = 5$ into the general solution and find that

$$\begin{aligned} -5 + e^0 + 0 &= C \\ C &= -4. \end{aligned}$$

Thus the particular solution is

$$-y + e^{ty} + t = -4.$$