MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Test 2	**MATH 2260**	Spring 2019

SOLUTIONS

1. First we rewrite the equation as

$$\frac{dy}{dt} + \frac{1}{t\cos(t)}y = \frac{1}{2t-1}.$$

This is a first-order linear equation with coefficients $p(t) = \dfrac{1}{t\cos(t)}$ and $g(t) = \dfrac{1}{2t-1}$. The discontinuities in $p(t)$ will occur when $t = 0$ and when $\cos(t) = 0$, that is, for $t = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. The only discontinuity in $g(t)$ occurs when $2t - 1 = 0$, that is, for $t = \frac{1}{2}$.

[3] (a) Since $t = 1$ falls between the discontinuities at $t = \frac{1}{2}$ and $t = \frac{\pi}{2}$, the interval of definition is $\frac{1}{2} < t < \frac{\pi}{2}$.

[3] (b) Since $t = -1$ falls between the discontinuities at $t = -\frac{\pi}{2}$ and $t = 0$, the interval of definition is $-\frac{\pi}{2} < t < 0$.

[8] 2. The fixed points occur at any value of y for which $\frac{dy}{dt} = 0$, namely $y = 0$, $y = 1$, $y = 3$ and $y = 6$. In the case of $y = 0$ and $y = 3$, $\frac{dy}{dt} < 0$ to their left and $\frac{dy}{dt} > 0$ to their right, so solutions that start near these fixed points are moving away from them. Thus $y = 0$ and $y = 3$ are <u>unstable</u>. The opposite is true of $y = 1$ and $y = 6$, so they are (asymptotically) <u>stable</u>.

If $y(0) = 2$ then y falls between the stable fixed point at $y = 1$ and the unstable fixed point at $y = 3$. Thus $\lim_{t\to\infty} y = 1$.

If $y(0) = 4$ then y falls between the unstable fixed point at $y = 3$ and the stable fixed point at $y = 6$. Thus $\lim_{t\to\infty} y = 6$.

[4] 3. (a) The characteristic equation is

$$9r^2 - 4 = 0 \quad\Longrightarrow\quad r^2 = \frac{4}{9} \quad\Longrightarrow\quad r = \pm\frac{2}{3}.$$

Hence the general solution is

$$y = C_1 e^{\frac{2}{3}t} + C_2 e^{-\frac{2}{3}t}.$$

[4] (b) The characteristic equation is

$$r^2 - 8r + 16 = 0 \quad\Longrightarrow\quad (r-4)^2 = 0 \quad\Longrightarrow\quad r = 4.$$

Since this is a double root, the general solution is

$$y = C_1 e^{4t} + C_2 t e^{4t}.$$

[4] 4. If $y = 3\cos(7t)$ is a solution then so too is any multiple of $\cos(7t)$. Hence the roots of the characteristic equation must have the form $r = \lambda \pm i\mu$ for $\lambda = 0$ and $\mu = 7$. Thus $r = \pm 7i$ and so the characteristic equation must be

$$(r - 7i)(r + 7i) = 0 \quad \Longrightarrow \quad r^2 - 49i^2 = 0 \quad \Longrightarrow \quad r^2 + 49 = 0.$$

This corresponds to the equation

$$\frac{d^2y}{dt^2} + 49y = 0.$$

[9] 5. (a) We assume that $y = v(t)t^3$. Then

$$\frac{dy}{dt} = \frac{dv}{dt}t^3 + 3v(t)t^2 \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2v}{dt^2}t^3 + 6\frac{dv}{dt}t^2 + 6v(t)t.$$

Substituting this into the equation, we obtain

$$\begin{aligned} t^2\left[\frac{d^2v}{dt^2}t^3 + 6\frac{dv}{dt}t^2 + 6v(t)t\right] - 6v(t)t^3 &= 0 \\ \frac{d^2v}{dt^2}t^5 + 6\frac{dv}{dt}t^4 &= 0 \\ \frac{d^2v}{dt^2}t + 6\frac{dv}{dt} &= 0. \end{aligned}$$

Let $u = \frac{dv}{dt}$ so $\frac{du}{dt} = \frac{d^2v}{dt^2}$. Then we have

$$\begin{aligned} \frac{du}{dt}t + 6u &= 0 \\ \int \frac{1}{u}\,du &= -6\int \frac{1}{t}\,dt \\ \ln(u) &= -6\ln(t) + C_2 \\ u &= C_2 t^{-6} \\ \frac{dv}{dt} &= C_2 t^{-6} \\ v(t) &= C_2 \int t^{-6}\,dt \\ &= C_2 t^{-5} + C_1. \end{aligned}$$

This means that the general solution is

$$y = [C_2 t^{-5} + C_1]t^3 = C_1 t^3 + C_2 t^{-2},$$

and so a distinct second solution is $y_2 = t^{-2}$.

[3] (b) The Wronskian is given by

$$\begin{aligned} W(t) &= y_1\frac{dy_2}{dt} - y_2\frac{dy_1}{dt} \\ &= t^3(-2t^{-3}) - t^{-2}(3t^2) \\ &= -2 - 3 \\ &= -5, \end{aligned}$$

which is not identically zero. Hence $\{t^3, t^{-2}\}$ forms a fundamental set of solutions to the equation.