MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 | **MATH 2260** | SPRING 2019

SOLUTIONS

[10] 1. We have

$$M(t,y) = 5ty + 4y^2 \quad \text{and} \quad N(t,y) = t^2 + 2ty$$

so

$$\frac{\partial M}{\partial y} = 5t + 8y \quad \text{and} \quad \frac{\partial N}{\partial t} = 2t + 2y.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, the equation is not exact.

However, note that

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) = \frac{3t + 6y}{t^2 + 2ty} = \frac{3(t+2y)}{t(t+2y)} = \frac{3}{t},$$

which is a function of t only. Hence an appropriate integrating factor is

$$\mu = e^{\int \frac{3}{t}\, dt} = e^{3\ln(t)} = t^3.$$

Multiplying through by the integrating factor, the ODE becomes

$$(5t^4y + 4t^3y^2) + (t^5 + 2t^4y)\frac{dy}{dt} = 0$$

so

$$M^*(t,y) = 5t^4y + 4t^3y^2 \quad \text{and} \quad N^*(t,y) = t^5 + 2t^4y.$$

Now

$$\frac{\partial M^*}{\partial y} = 5t^4 + 8t^3y = \frac{\partial N^*}{\partial t},$$

so the equation has been made exact. Thus there exists a function $\psi(t,y)$ such that

$$\frac{\partial \psi}{\partial t} = 5t^4y + 4t^3y^2 \quad \text{and} \quad \frac{\partial \psi}{\partial y} = t^5 + 2t^4y.$$

This means that

$$\begin{aligned} \psi(t,y) &= \int (5t^4y + 4t^3y^2)\, dt \\ &= t^5y + t^4y^2 + C(y) \end{aligned}$$

$$\frac{\partial \psi}{\partial y} = t^5 + 2t^4y + C'(y) = t^5 + 2t^4y$$

$$C'(y) = 0$$

$$C(y) = C.$$

Finally, the general solution must be $\psi(t,y) = C$, that is,

$$t^5y + t^4y^2 = C.$$

[30] 2. (a) This equation is separable. It can be written

$$(t^2 - 3t + 2)\frac{dy}{dt} = ty$$
$$\int \frac{1}{y}\,dy = \int \frac{t}{t^2 - 3t + 2}\,dt.$$

To evaluate the integral on the right, we use partial fractions:

$$\frac{t}{t^2 - 3t + 2} = \frac{t}{(t-2)(t-1)} = \frac{A}{t-2} + \frac{B}{t-1}$$
$$t = A(t-1) + B(t-2).$$

When $t = 2$ we have $2 = A(1)$ so $A = 2$. When $t = 1$ we have $1 = B(-1)$ so $B = -1$. Thus

$$\int \frac{1}{y}\,dy = \int \left(\frac{2}{t-2} - \frac{1}{t-1}\right)\,dt$$
$$\ln(y) = 2\ln(t-2) - \ln(t-1) + \ln(C)$$
$$= \ln\left(\frac{C(t-2)^2}{t-1}\right)$$
$$y = \frac{C(t-2)^2}{t-1}.$$

Finally, since $y(3) = 8$ we have

$$y(3) = \frac{C}{2} = 8 \quad \Longrightarrow \quad C = 16.$$

Thus the particular solution is

$$y = \frac{16(t-2)^2}{t-1}.$$

(b) This equation is linear (but not separable) so we first rewrite it as

$$\frac{dy}{dt} - \frac{3}{t}y = t^4\cos(t).$$

Hence $p(t) = -\frac{3}{t}$ and therefore an appropriate integrating factor is

$$\mu = e^{-3\int \frac{1}{t}\,dt} = e^{-3\ln(t)} = t^{-3}.$$

Multiplying through by t^{-3} we obtain

$$\begin{aligned} t^{-3}\frac{dy}{dt} - 3t^{-4}y &= t\cos(t) \\ \frac{d}{dt}[t^{-3}y] &= t\cos(t) \\ t^{-3}y &= \int t\cos(t)\,dt \\ &= t\sin(t) + \cos(t) + C \\ y &= t^4\sin(t) + t^3\cos(t) + Ct^3. \end{aligned}$$

Note that the integral on the right can be evaluated using integration by parts. Alternatively, we can rewrite the given equation as

$$-3y - t^5\cos(t) + t\frac{dy}{dt} = 0,$$

for which

$$M(t,y) = -3y - t^5\cos(t) \quad \text{and} \quad N(t,y) = t.$$

Then

$$\frac{\partial M}{\partial y} = -3 \quad \text{and} \quad \frac{\partial N}{\partial t} = 1$$

and since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial t}$ we know that the equation is not exact. However,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) = -\frac{4}{t}$$

and so an appropriate integrating factor is

$$\mu = e^{-4\int \frac{1}{t}\,dt} = e^{-4\ln(t)} = t^{-4}.$$

Multiplying through by μ, the equation becomes

$$-3t^{-4}y = t\cos(t) + t^{-3}\frac{dy}{dt} = 0,$$

which can be verified to be exact. Hence

$$\psi(t,y) = \int t^{-3}\,dy = t^{-3}y + C(t)$$

and so

$$\begin{aligned} \frac{\partial \psi}{\partial t} = -3t^{-4}y + C'(t) &= -3t^{-4}y - t\cos(t) \\ C'(t) &= -t\cos(t) \\ C(t) &= -\int t\cos(t)\,dt \\ &= -t\sin(t) - \cos(t) + C \end{aligned}$$

via integration by parts. Hence the solution is $\psi(t,y) = C$, that is,

$$t^{-3}y - t\sin(t) - \cos(t) = C.$$

Since the equation was linear, we must solve for y and conclude that

$$y = t^4\sin(t) + t^3\cos(t) + Ct^3.$$

(c) We can rewrite this equation as

$$\frac{dy}{dt} = \frac{t^2+ty+y^2}{t^2}.$$

If we let $f(t,y) = \dfrac{t^2+ty+y^2}{t^2}$ then, for any constant k,

$$\begin{aligned} f(kt,ky) &= \frac{(kt)^2+(kt)(ky)+(ky)^2}{(kt)^2} \\ &= \frac{k^2t^2+k^2ty+k^2y^2}{k^2t^2} \\ &= \frac{k^2(t^2+ty+y^2}{k^2t^2} \\ &= \frac{t^2+ty+y^2}{t^2} \\ &= f(t,y) \end{aligned}$$

and so this equation is homogeneous.
Now we let $y = vt$ so $\dfrac{dy}{dt} = \dfrac{dv}{dt}t + v$. The equation becomes

$$\begin{aligned} \frac{dv}{dt}t + v &= \frac{t^2+vt^2+v^2t^2}{t^2} \\ &= 1+v+v^2 \\ \frac{dv}{dt}t &= 1+v^2 \\ \int \frac{1}{1+v^2}\,dv &= \int \frac{1}{t}\,dt \\ \arctan(v) &= \ln(t)+C \\ \arctan\left(\frac{y}{t}\right) &= \ln(t)+C. \end{aligned}$$

Although the equation is non-linear, it is straightforward to solve for y, and thus the solution is

$$y = t\tan(\ln(t)+C).$$

(d) This is a Bernoulli equation with $n = 3$, so we first rewrite the ODE as

$$y^{-3}\frac{dy}{dt} + \frac{1}{\sqrt{t}}y^{-2} = \frac{1}{\sqrt{t}}.$$

Now we let $v = y^{-2}$ so $-\frac{1}{2}\frac{dv}{dt} = y^{-3}\frac{dy}{dt}$. The equation becomes

$$\begin{aligned} -\frac{1}{2}\frac{dv}{dt} + \frac{1}{\sqrt{t}}v &= \frac{1}{\sqrt{t}} \\ \frac{dv}{dt} - \frac{2}{\sqrt{t}}v &= -\frac{2}{\sqrt{t}}. \end{aligned}$$

Since this is a linear equation with $p(t) = -\frac{2}{\sqrt{t}}$ an appropriate integrating factor is

$$\mu = e^{-2\int \frac{1}{\sqrt{t}}\,dt} = e^{-4\sqrt{t}}.$$

Multiplying the equation through by μ yields

$$\begin{aligned} e^{-4\sqrt{t}}\frac{dv}{dt} - \frac{2}{\sqrt{t}}e^{-4\sqrt{t}}v &= -\frac{2}{\sqrt{t}}e^{-4\sqrt{t}} \\ \frac{d}{dt}\left[e^{-4\sqrt{t}}v\right] &= -\frac{2}{\sqrt{t}}e^{-4\sqrt{t}} \\ e^{-4\sqrt{t}}v &= -2\int \frac{1}{\sqrt{t}}e^{-4\sqrt{t}}\,dt. \end{aligned}$$

In order to evaluate the integral on the right we let $u = \sqrt{t}$ so $du = -\frac{2}{\sqrt{t}}\,dt$. Now we have

$$\begin{aligned} e^{-4\sqrt{t}}v &= \int e^u\,du \\ &= e^u + C \\ &= e^{-4\sqrt{t}} + C \\ v &= 1 + Ce^{4\sqrt{t}} \\ y^{-2} &= 1 + Ce^{4\sqrt{t}}. \end{aligned}$$

Alternatively, we could use the fact that this equation is separable since it can be written

$$\begin{aligned} \frac{dy}{dt} &= \frac{y^3 - y}{\sqrt{t}} \\ \int \frac{1}{y^3 - y}\,dy &= \int \frac{1}{\sqrt{t}}\,dt. \end{aligned}$$

The integral on the left requires partial fractions, for which we have

$$\frac{1}{y^3 - y} = \frac{1}{y(y-1)(y+1)} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+1}$$
$$1 = A(y-1)(y+1) + By(y+1) + Cy(y-1).$$

When $y = 0$ we have $1 = A(-1)$ so $A = -1$. When $y = 1$ we have $1 = B(2)$ so $B = \frac{1}{2}$. When $y = -1$ we have $1 = C(2)$ so $C = \frac{1}{2}$ as well. Thus we obtain

$$\int \left[-\frac{1}{y} + \frac{\frac{1}{2}}{y-1} + \frac{\frac{1}{2}}{y+1} \right] dy = \int \frac{1}{\sqrt{t}}\, dt$$
$$-\ln(y) + \frac{1}{2}\ln(y-1) + \frac{1}{2}\ln(y+1) = 2\sqrt{t} + C$$
$$\ln\left(\frac{\sqrt{(y-1)(y+1)}}{y} \right) = 2\sqrt{t} + C$$
$$\frac{\sqrt{y^2-1}}{y} = Ce^{2\sqrt{t}}.$$

These solutions may not appear to be equivalent, but note that we can further rewrite the latter solution by squaring both sides to get

$$\frac{y^2-1}{y^2} = Ce^{4\sqrt{t}}$$
$$y^2 - 1 = Ce^{4\sqrt{t}}y^2$$
$$y^2 + Ce^{4\sqrt{t}}y^2 = 1$$
$$1 + Ce^{4\sqrt{t}} = y^{-2},$$

as before.