MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 4 | **MATH 2260** | SPRING 2019

Due: Wednesday, June 19th, 2019 at 1:00pm. SHOW ALL WORK.

1. Determine the interval of definition for the solution of the given linear initial value problem. Do NOT solve the equation.

   (a) $t^3 \dfrac{dy}{dt} - (t+1)y = \dfrac{1}{t-4}, \quad y(2) = -3$

   (b) $t^3 \dfrac{dy}{dt} - (t+1)y = \dfrac{1}{t-4}, \quad y(-3) = 2$

   (c) $\dfrac{dy}{dt} + \tan(t)y = 0, \quad y(\pi) = 0$

   (d) $(9 - t^2)\dfrac{dy}{dt} - y = \cos(3t), \quad y(0) = -1$

2. Indicate a rectangle (that is, an interval of t-values and an interval of y-values) in which the requirements of the theorem on existence and uniqueness are satisfied for the non-linear initial value problem

$$1 - \sin(t) + y^2 + (ty - 2y + 4t - 8)\frac{dy}{dt} = 0$$

   with the given initial condition. If no such rectangle exists, explain why not. Do NOT solve the equation.

   (a) $y(0) = 3$ (b) $y(5) = -5$ (c) $y(1) = -4$

3. Consider a tank which initially contains V litres of water and Q_0 kilograms of salt. Suppose that a new mixture of brine at a concentration of k kg per litre is poured into the tank, the contents of the vat are thoroughly mixed, and the contents of the tank are drained at the same rate. Unlike the model we studied in class, however, now assume that the rate of inflow and outflow is proportional to the length of time since the mixing began.

   (a) Set up a new differential equation to model this situation.

   (b) Solve the equation, and determine what effect (if any) this change to the rate of inflow/outflow has on the long-term behaviour of the salt concentration in the tank.

PLEASE TURN OVER

4. Recall that if a population y grows logistically then it can be described by the autonomous differential equation

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right). \tag{1}$$

In class, we considered the case where y is a population of codfish and modified Equation (1) to incorporate the effects of fishing. We assumed that the codfish are harvested at a constant rate F, so that the rate of change of the population is given by

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) - F. \tag{2}$$

Suppose instead that the rate of harvesting is proportional to the size of the population. In other words, the amount of fishing goes up when more codfish are available, but is reduced when the population is smaller. (This is known as **effort harvesting**.) The equation which describes the model now becomes

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) - Fy, \tag{3}$$

where F is now a constant of proportionality.

(a) Find the fixed points of Equation (3). How do these compare to the fixed points of Equation (1)?

(b) Draw a graph of $\frac{dy}{dt}$ vs. y (for a small value of F) and use it to determine the stability of each fixed point. How does the stability of each fixed point compare with the stability of the fixed points of Equation (1)?

(c) If F is made too large, the codfish population will perish regardless of how many fish are present initially. Find the value F^* (in terms of r and/or K) for which the codfish population is viable when $F < F^*$, but not viable when $F > F^*$.