8. WE ASSUME

$$y = u(t)t + v(t)t\ln(t)$$

$$\frac{dy}{dt} = \frac{du}{dt}t + u(t) + \frac{dv}{dt}t\ln(t) + v(t)\ln(t) + v(t)$$

WE ASSUME

$$\frac{du}{dt}t + \frac{dv}{dt}t\ln(t) = 0$$

SO $$\frac{dy}{dt} = u(t) + v(t)\ln(t) + v(t)$$

$$\frac{d^2y}{dt^2} = \frac{du}{dt} + \frac{dv}{dt}\ln(t) + v(t)t^{-1} + \frac{dv}{dt}$$

WE SUBSTITUTE INTO THE ODE AND OBTAIN

$$t^2\left[\frac{du}{dt} + \frac{dv}{dt}\ln(t) + v(t)t^{-1} + \frac{dv}{dt}\right] - t\left[u(t) + v(t)\ln(t) + v(t)\right] + \left[u(t)t + v(t)t\ln(t)\right] = \ln(t)$$

$$t^2\frac{du}{dt} + t^2\ln(t)\frac{dv}{dt} + t^2\frac{dv}{dt} = \ln(t)$$

WE MUST SOLVE THE SYSTEM

$$\begin{cases} t\frac{du}{dt} + t\ln(t)\frac{dv}{dt} = 0 \\ t^2\frac{du}{dt} + t^2\ln(t)\frac{dv}{dt} + t^2\frac{dv}{dt} = \ln(t) \end{cases} \quad \rightarrow \quad \frac{du}{dt} = -\ln(t)\frac{dv}{dt}$$

NOW WE HAVE

$$t^2\left[-\ln(t)\frac{dv}{dt}\right] + t^2\ln(t)\frac{dv}{dt} + t^2\frac{dv}{dt} = \ln(t)$$

$$t^2\frac{dv}{dt} = \ln(t)$$

$$\frac{dv}{dt} = t^{-2}\ln(t)$$

THEN

$$v(t) = \int t^{-2}\ln(t)\,dt$$

$w = \ln(t) \quad dv = t^{-2}dt$

$dw = \frac{1}{t}dt \quad v = -\frac{1}{t}$

$$= -t^{-1}\ln(t) + \int t^{-2}dt$$

$$= -t^{-1}\ln(t) - t^{-1} + C_2$$

NEXT,

$$\frac{du}{dt} = -\ln(t)\left[t^{-2}\ln(t)\right]$$

$$= -t^{-2}\ln^2(t)$$

$$u(t) = \int \left(-t^{-2}\ln^2(t)\right)dt$$

$w = \ln^2(t) \quad dv = -t^{-2}dt$

$dw = 2t^{-1}\ln(t)dt \quad v = \frac{1}{t}$

$$= t^{-1}\ln^2(t) - \int 2t^{-2}\ln(t)\,dt$$

$w = \ln(t) \quad dv = 2t^{-2}dt$

$dw = \frac{1}{t}dt \quad v = -\frac{2}{t}$

$$= t^{-1}\ln^2(t) - \left[-2t^{-1}\ln(t) + \int 2t^{-2}dt\right]$$

$$= t^{-1}\ln^2(t) + 2t^{-1}\ln(t) - 2\left[-t^{-1}\right] + C_1$$

$$= t^{-1}\ln^2(t) + 2t^{-1}\ln(t) + 2t^{-1} + C_1$$

THE GENERAL SOL'N IS

$$y = \left[t^{-1}\ln^2(t) + 2t^{-1}\ln(t) + 2t^{-1} + C_1\right]t + \left[-t^{-1}\ln(t) - t^{-1} + C_2\right]t\ln(t)$$

$$= C_1 t + C_2 t\ln(t) + \ln^2(t) + 2\ln(t) + 2 - \ln^2(t) - \ln(t)$$

$$= C_1 t + C_2 t\ln(t) + \ln(t) + 2$$